

## Phase Diagram of a 2D Vertex Model

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The two-dimensional (2D) vertex model is one of the most intensively studied systems in statistical physics.<sup>1)</sup> Analytic expressions of thermodynamic and correlation functions are known in the parameter area where Yang-Baxter relation is satisfied.<sup>1,2)</sup> The phase diagram outside the solvable area is, however, not known completely.

As a small step to explore the phase diagram of the 16-vertex model, we investigate its subset which allows 7 vertex configurations shown in Fig. 1, where thick and thin lines, respectively, correspond to the spin states 0 and 1 on the bond.<sup>3)</sup> The Yang-Baxter relation is not satisfied since there is no configuration where two 1s touch such as (1100) and (0110).

If we interpret 0 and 1, respectively, as absence and presence of adsorbed atoms on the solid surface, the condition in Fig. 1 is equivalent to setting the minimum distance between adsorbed atoms as the lattice constant. In other word, the model is equivalent to the hard square model<sup>1)</sup> on the diagonal lattice.

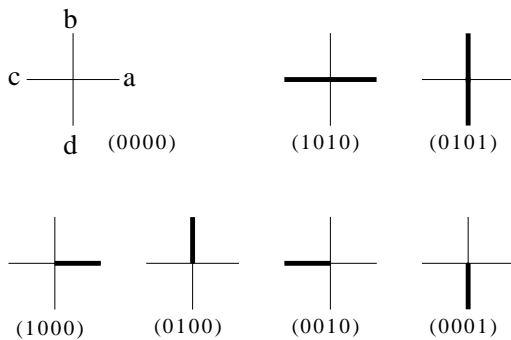


Fig. 1. Allowed vertex configurations, that are  $(abcd) = (0000), (1010), (0101), (1000), (0100), (0010),$  and  $(0001)$ .

An alternative interpretation of the model is, as it has been done for the 6-vertex model, to regard it as 2D lattice polymer on the solid surface, where 0 (1) represents absence (presence) of a unit monomer on the bond. According to the allowed configurations in Fig. 1, only straight polymers are allowed to exist. As shown in Fig. 2, it is easily imagined that ordered phase appears when the polymer length is long or its density is high, and that disordered phase appears under the opposite conditions. The aim of this short note is to draw the

phase boundary of this order-disorder transition.

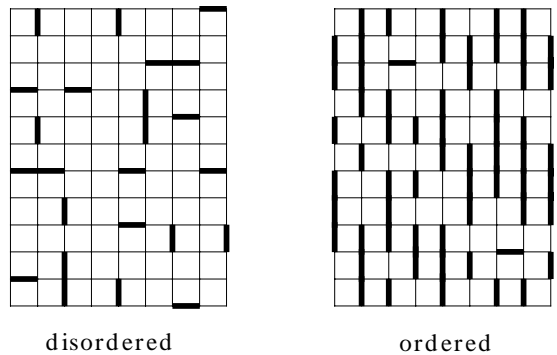


Fig. 2. Disordered state and the ordered state.

Let us parameterize the vertex model from the view point of the lattice polymer. For simplicity, we assume that the local Boltzmann weights are symmetric under the right angle rotations. Each vertex weight is then expressed as

$$\begin{aligned} W(0000) &= \exp(0) = 1 \\ W(1010) &= W(0101) = \exp(K) \\ W(1000) &= W(0100) = W(0010) \\ &= W(0001) = \exp(K/2 + B) \end{aligned} \quad (1)$$

using two parameters  $K \equiv -\beta U$  and  $B \equiv -\beta V$ , where  $U$  is the energy per length of the line polymer and  $V$  is its boundary energy; the Boltzmann weight for a straight line of length  $N$  is  $\exp(NK + 2B)$ .

An appropriate order parameter for the order-disorder transition in Fig. 2 is

$$\begin{aligned} M &= P(1010) + \frac{P(1000) + P(0010)}{2} \\ &\quad - P(0101) - \frac{P(0100) + P(0001)}{2}, \end{aligned} \quad (2)$$

where  $P(abcd)$  is the absolute probability to observe the local configuration  $(abcd)$ . We calculate  $M$  at the center of square systems with linear dimension  $L$  using the CTMRG method,<sup>4)</sup> which is a variant of the density matrix renormalization group (DMRG)<sup>5,6)</sup> applied to 2D classical systems,<sup>7)</sup> where  $L$  is chosen to be sufficiently larger than the correlation length. To weakly stabilize the order in the horizontal direction, we impose fixed

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boundary conditions where all the spins on the vertical sides of the square are 1 and those on the horizontal sides are 0. This boundary condition makes the corner transfer matrix asymmetric, therefore we use an asymmetric extension of the CTMRG method.

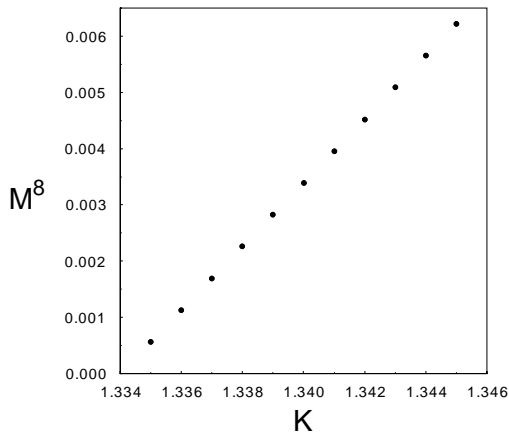


Fig. 3. The eighth power of the order parameter ( $= M^8$ ) with respect to  $K$  when  $B = 0$ .

The phase boundary is determined from the observation of the order parameter  $M$ . Figure 3 shows the  $K$ -dependence of  $M^8$  when  $B = 0$ . It is clear that  $M^8$  is a linear function of  $K$ , and we obtain the critical point  $K_c = 1.334$ . Also for the cases where  $B \neq 0$  we have checked that  $M^8$  is linear in  $K$ ; the critical index  $\beta$  of this system is  $1/8$ .

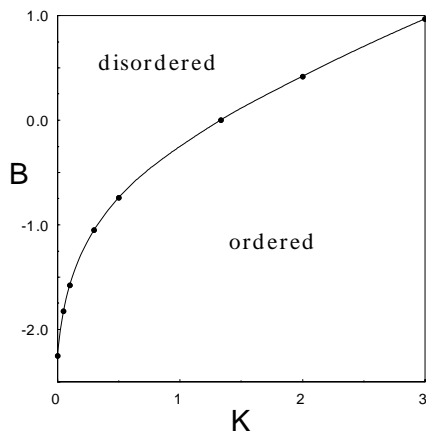


Fig. 4. Phase diagram in the  $K$ - $B$  plane.

We thus obtained the phase boundary in the  $K$ - $B$  plane shown in Fig. 4. The system is always disordered in the negative  $K$  region, therefore we only show the positive  $K$  region. The boundary starts from  $(K, B) = (0_+, -2.25)$ , it passes  $(K, B) = (1.334, 0)$ , and then approaches the line  $B = (K - 1.0)/2$ ; we have not identified the reason why the line has the offset of amount  $-1.0/2 = -0.5$ .

It is worth obtaining another critical index of this

model, which is related to the density defined by

$$n = P(1010) + \frac{P(1000) + P(0010)}{2} + P(0101) + \frac{P(0100) + P(0001)}{2}. \quad (3)$$

Figure 5 shows the  $K$  dependence of  $n - n_c$  when  $B = 0$ , where  $n_c = 0.736$  is the density at the phase boundary  $K_c = 1.334$ . It is apparent that  $n - n_c$  is proportional to  $K - K_c$ . Since  $n$  is related to the energy parameter  $U$ , the exponent  $\alpha$  is equal to 0.

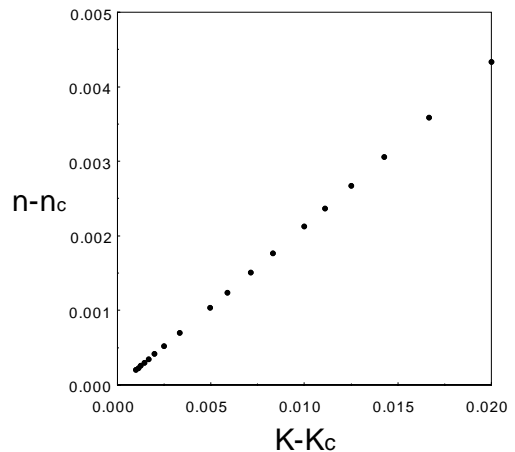


Fig. 5. Density in Eq. 3 near the transition point  $K_c = 1.334$  when  $B = 0$ , where  $n_c = 0.736$ .

In conclusion we have studied a symmetric vertex model which allows 7 vertex configurations shown in Fig. 1, and obtained its phase diagram. The critical indices of this model, that are  $\beta = 1/8$  and  $\alpha = 0$ , are the same as those of the Ising model. To include additional configurations, such as (1111) and (1110), that may cause the percolation transition, is our next subject toward the complete understanding of the 16-vertex model.

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