Phase Transition under Hyperbolic Geometry
... from classical to quantum ...

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Hyperbolic Lattice

Spherical Lattice

http://quattro.phys.sci.kobe-u.ac.jp/Hyperbolic/Hyperbolic.html
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Ising Model on the (5,4) Lattice

\[ H = - J \sum \sigma_i \sigma_j \quad (i \text{ and } j \text{ are NN sites}) \]

Commuting Transfer Matrix .... not known yet.

Monte Carlo Simulation .... rather hard

DMRG / CTMRG .... Yes, Applicable (Skip the detail.)
Recursive Structure of the (5,4) lattice

In general, (q,p=even) lattice has the same property.

Half `row' transfer matrix $P$

(after taking configuration sum for spins inside, leaving those on the boundary)

Corner transfer matrix $C$
(See http://quattro.phys.sci.kobe-u.ac.jp/Hyperbolic/Hyperbolic.html.)
The Mean-Field Universality is observed.

Transition is NOT Critical.

\[ \text{Column-to-Column Transfer Matrix } P^*P \]

Fig. 2. Correlation Length $\xi$ with respect to the temperature $k_B T/J$.

Fig. 3. Entanglement entropy of the MPS in Eq. (2.12).
NNN Ising model (J1-J2 Ising)

\[ \mathcal{H} = -J_1 \sum_{\langle ij \rangle = \text{NN}} \sigma_i \sigma_j + J_2 \sum_{\langle ik \rangle = \text{NNN}} \sigma_i \sigma_k \]

![Graph showing the phase diagram of the NNN Ising model](image)

- **Paramagnetic**
- **Ferromagnetic**
- **Tricritical (??) Point!**

For purpose of obtaining brief insight of the phase transitions on hyperbolic lattices need not always be mean-field like. Recent numerical studies have supported this conclusion.
q-state Clock Models

\[ W(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = \prod_{i=1}^{5} \exp \left\{ \frac{J \cos (\theta_i - \theta_{i+1})}{2 k_B T} \right\} \]

\[ \mathcal{H} = -J \sum_{\langle ij \rangle} \cos (\theta_i - \theta_j) \]

q=3: First Order Transition
q>4: Mean-Field 2nd Order
Toward Hyperbolic 1+1 dimensional Space

... if one discretize the space, one would obtain non-uniform Hamiltonian ...

\[ H = \sum_j \cosh(j\lambda) \, h_{j,j+1} \]

Is the ground-state of such a Hamiltonian uniform???

(*) one can consider the same sort of deformations on Sphere, Projective plane, Klein tube, etc.
Transverse Field Ising Model under Hyperbolic Deformation

\[ H^c(\lambda) = -J \sum_j \cosh[j\lambda] \sigma_j^z \sigma_{j+1}^z - \Gamma \sum_j \cosh[(j - \frac{1}{2})\lambda] \sigma_j^x \]

Deep inside the system, the ground state is quite uniform.

Fig. 2. On-site transverse interaction \( \langle \Gamma \sigma_j^x \rangle \).

Correlation Length appears as the dumping rate of the boundary effect.

Correlation Length is inverse proportional to the deformation parameter.
Tips in DMRG:

Numerical Precision increases if one subtract the average of the Hamiltonian; this prevents the appearance of large matrix element in the block Hamiltonian. As a result, computation is stabilized.

We employ DMRG method\(^5\) for the numerical determination of the ground state. A direct application of the finite-system DMRG algorithm encounters a numerical instability, which is caused by the blow-up of the energy scale \(\cosh[j\lambda]\) with respect to \(|j|\). In order to stabilize the computation, we treat

\[
\begin{align*}
\tilde{H}^c_{P;L}(\lambda) & \equiv H^c_{P;L}(\lambda) - \langle H^c_{P;L}(\lambda) \rangle \\
\tilde{H}^c_{F;L}(\lambda) & \equiv H^c_{F;L}(\lambda) - \langle H^c_{F;L}(\lambda) \rangle
\end{align*}
\]

instead of \(H^c_{P;L}(\lambda)\) and \(H^c_{F;L}(\lambda)\) directly, where \(\langle \rangle\) denotes expectation value taken by the ground state. The smallest eigenvalue of both \(\tilde{H}^c_{P;L}(\lambda)\) and \(\tilde{H}^c_{F;L}(\lambda)\) is zero by definition. Since the ground state is not a priori known, the subtraction of Eqs. (2.3) is performed.

Imagine what happens if one repeat infinite system DMRG steps INFINITELY.
Energy CrossOver

Local Energy Density Operator: \[ h_{j,j+1} = -\left[ \frac{\Gamma}{2} \sigma_j^x + J \sigma_j^z \sigma_{j+1}^z + \frac{\Gamma}{2} \sigma_{j+1}^x \right] \]

There is no singularity at the transition point gamma = 1.

>>> this is the First Order Phase Transition >>>
and the paramagnetic condition when
romagnetic boundary condition is imposed when
which coincides the radius of curvature
tional to 1
the state that gives lowest value of
the bulk property of the system, it is natural to choose
sition. Since we have considered that
should be careful about the definition of the phase tran-
exist even in the large
lowest energy state alternates at the point
negligible, and for these cases the solid and dotted lines
evaluation of the ground-state energy. Figure 3 shows
served in the previous section would justify this way of
the local energy density of the bulk. The uniformity ob-
since subtraction of the boundary energy is not straight-

It should be noted that in a certain region of
-dependence is very small for
for

We show

(See

λ

1

0.97

L

=16

0.96

L

=32,

=24

=16

L

0.6

Γ

ξ

0.2

0.35

0.2

0.4

m=16

λ=1 +

λ=1/2 ×

λ=1/4

λ=1/8 ○

λ=1/16 △

0.15

0.4

0.3

0.25

0.2

0.15

0.1

0.05

0

0

0.2

0.4

0.6

0.8

1

1.2

Γ

λ

Spontaneous magnetization jumps at the transition point.

8-th power of the jump is proportional to the deformation parameter. The Ising Universality is detected, somehow.
Entanglement Entropy

also jumps at the transition point.

\[
\frac{1}{2} \cdot \frac{1}{6} \log \frac{1}{\lambda} + \frac{0.2475}{2}
\]

\[(B)-(A) \sim (1/2) \log 2\]

These are consistent with Ising Universality when the system size is restricted to $1/\lambda$.\ldots
Missing Link:

Classical (isotropic) Ising on the Hyperbolic Lattice shows 2nd order transition.

Quantum Ising under Hyperbolic Deformation shows 1st order transition.

One should construct the anisotropic limit.

**Fig. 2.** Correlation Length $\xi$ with respect to the temperature

**Fig. 3.** Entanglement entropy of the MPS in Eq. (2.12).
A Generalization: Spherical Deformation

N-site tight binding Hamiltonian

\[ \hat{H}_S = -t \sum_{\ell=1}^{N-1} \sin \frac{2\ell\pi}{N} \left( \hat{c}_\ell \hat{c}_{\ell+1} + \hat{c}_{\ell+1} \hat{c}_\ell \right) \]

Boundary effect on the bond energy disappears completely!

A system under Open Boundary Condition gives data as efficient as those under Periodic Boundary Condition, under the spherical deformation.
Conclusions

- Still there are many interest in 1D systems.
- Please memorize the term Hyperbolic Deformation and put it into Google.

Please visit the Web Pages

- [http://quattro.phys.sci.kobe-u.ac.jp/nishino_e.html](http://quattro.phys.sci.kobe-u.ac.jp/nishino_e.html) (My personal page)
- [http://quattro.phys.sci.kobe-u.ac.jp/Hyperbolic/Hyperbolic.html](http://quattro.phys.sci.kobe-u.ac.jp/Hyperbolic/Hyperbolic.html) (Hyperbolic)

(**) I probably miss about 20 percent of articles each year. Recently xxx.lanl.gov provides key word search for the whole part of the submitted papers. Day by day I search the Key Word, DMRG, and put the missing articles to the list. For those years 1992-2003 I completed the task.

The book “Density-Matrix Renormalization” ed. I Peschel et al (Springer 1999) is NOT on line. Is there any way to make it open access?