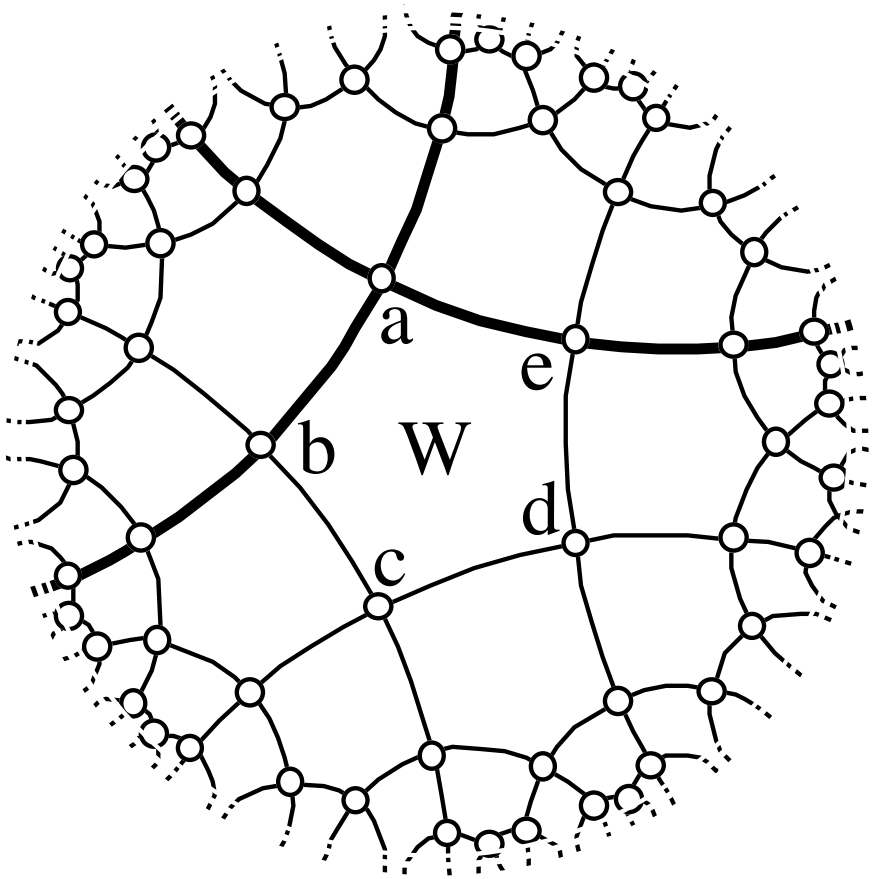
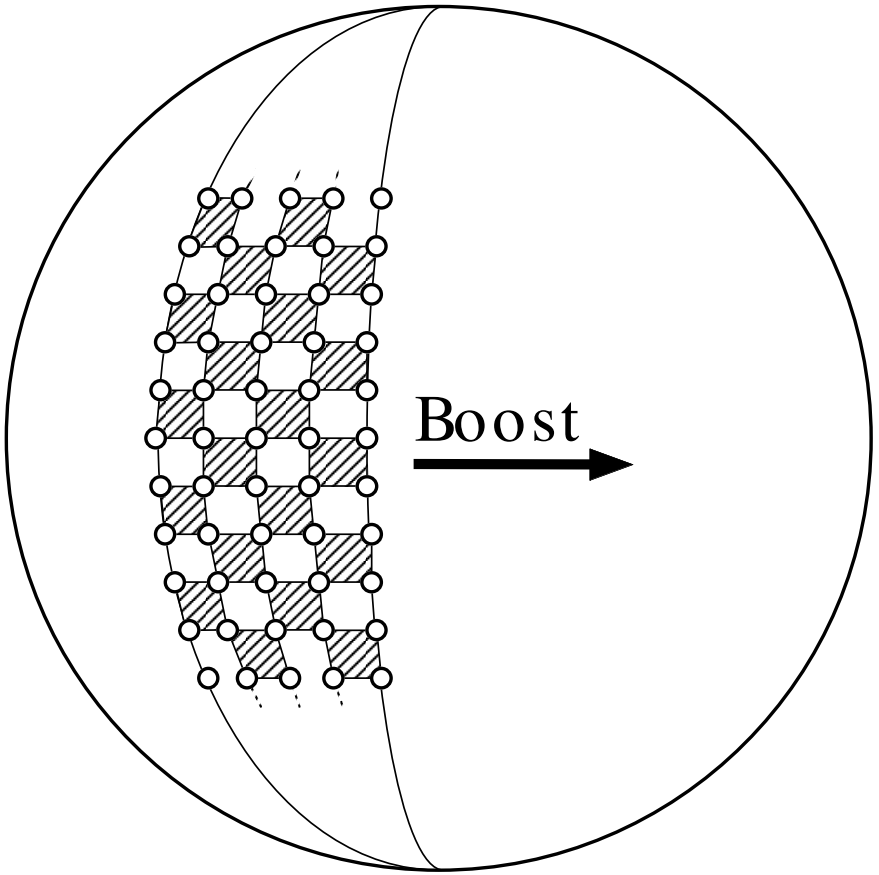
Phase Transition under Hyperbolic Geometry
... from classical to quantum ...

T. Nishino, A. Gendiar, H. Ueda, T. Iharagi



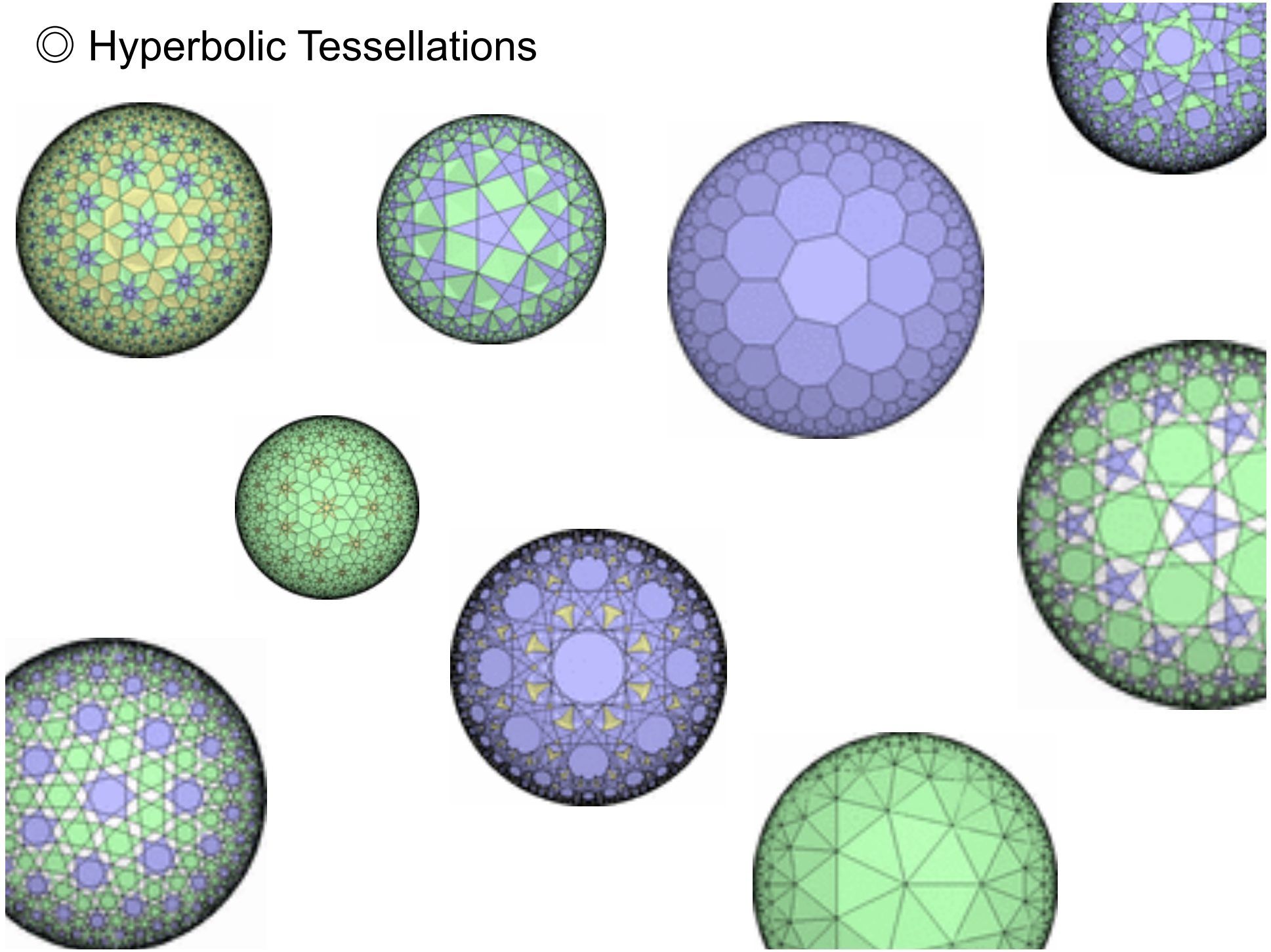
Hyperbolic Lattice



Spherical Lattice

http://quattro.phys.sci.kobe-u.ac.jp/Hyperbolic/Hyperbolic.html

◎ Hyperbolic Tessellations



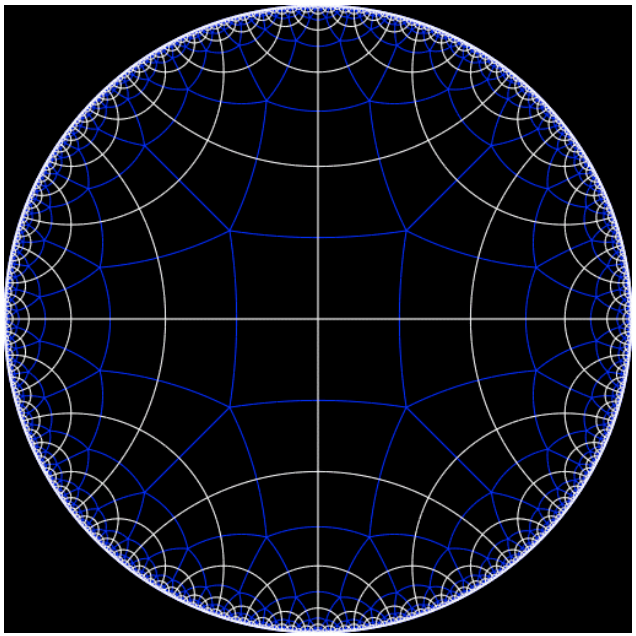
Ising Model on the (5,4) Lattice

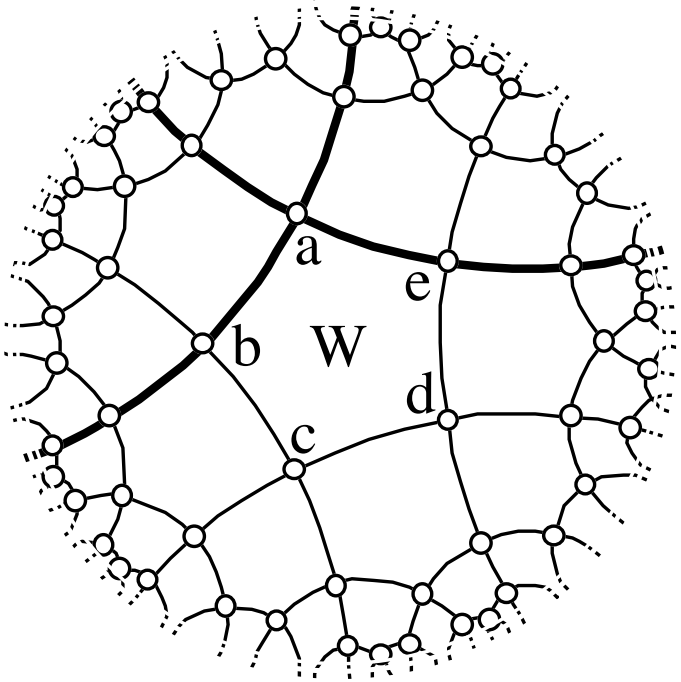
$$H = - J \sum \sigma_i \sigma_j \qquad (\text{ i and j are NN sites })$$

Commuting Transfer Matrix .... not known yet.

Monte Carlo Simulation .... rather hard

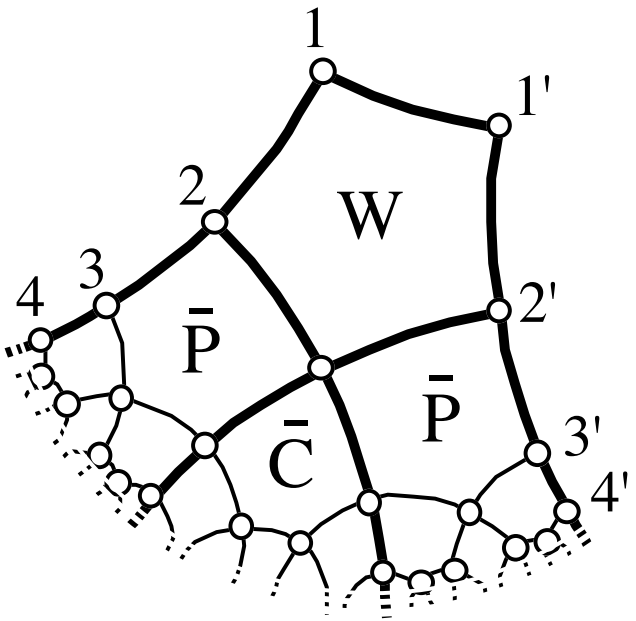
DMRG / CTMRG .... Yes, Applicable (Skip the detail.)



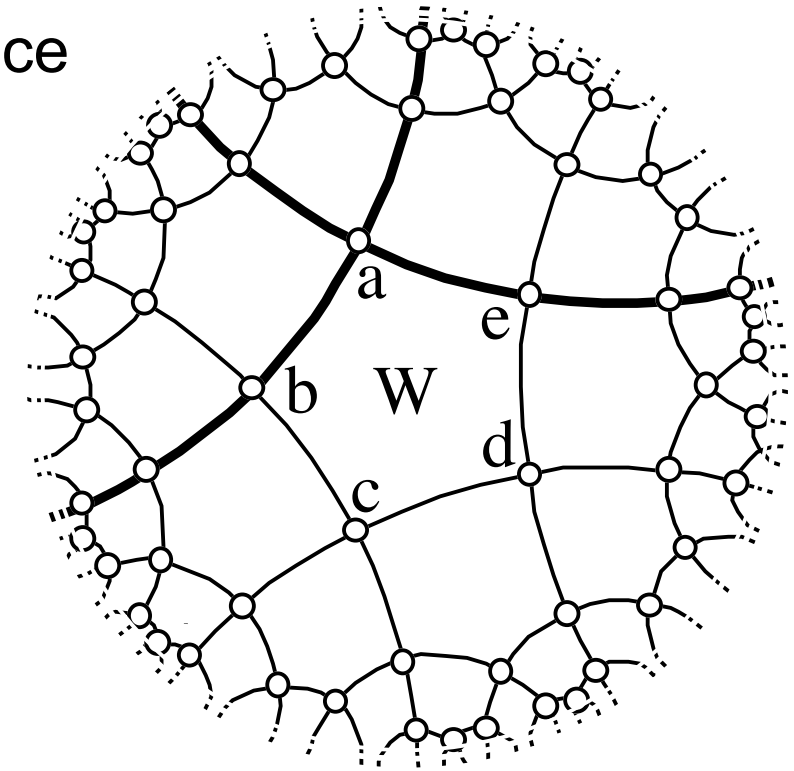


◎ Recursive Structure of the (5,4) lattice

in general, (q,p=even) lattice has the same property.

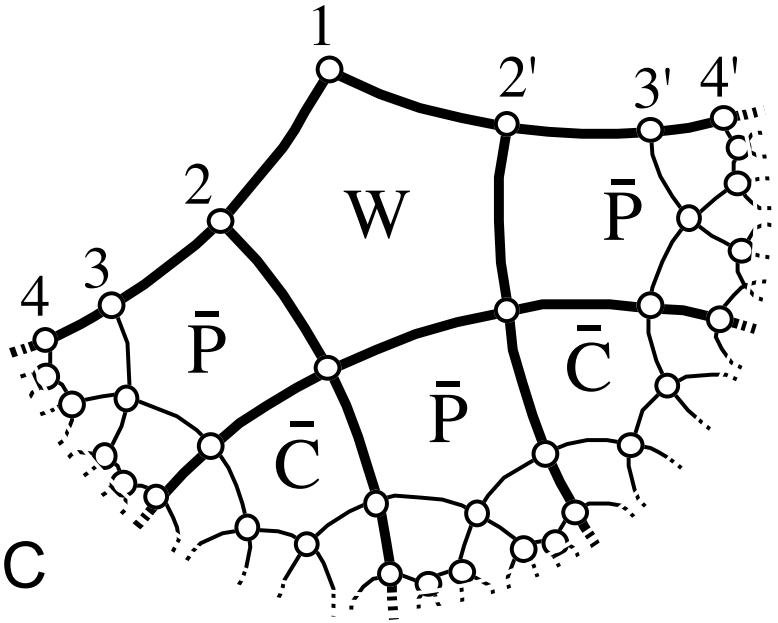


Structure of the Lattice



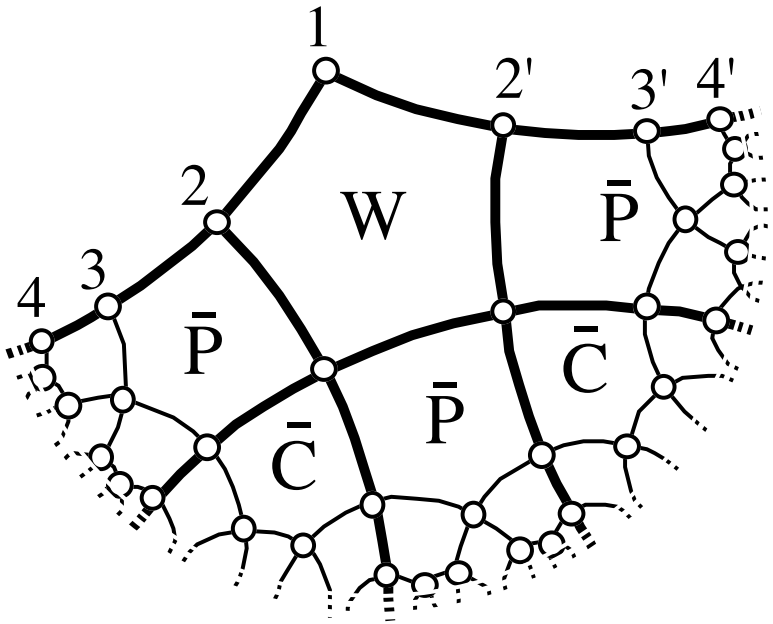
Half `row' transfer matrix P

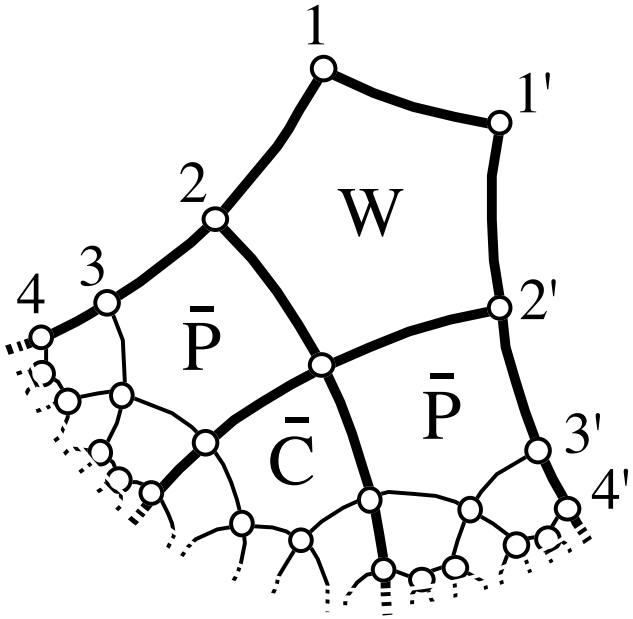
(after taking configuration sum for spins inside, leaving those on the boundary)

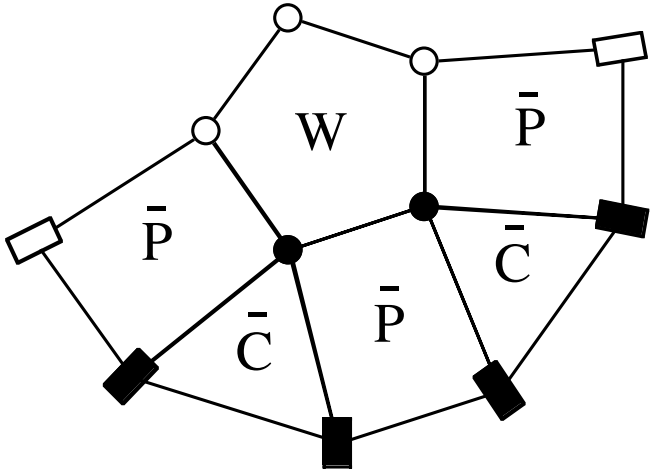


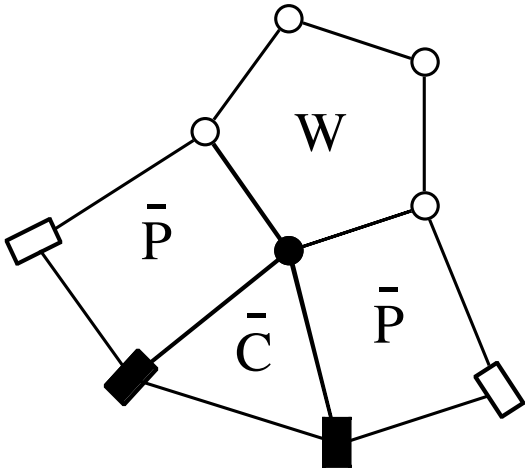
Corner transfer matrix C

(See http://quattro.phys.sci.kobe-u.ac.jp/Hyperbolic/Hyperbolic.html.)

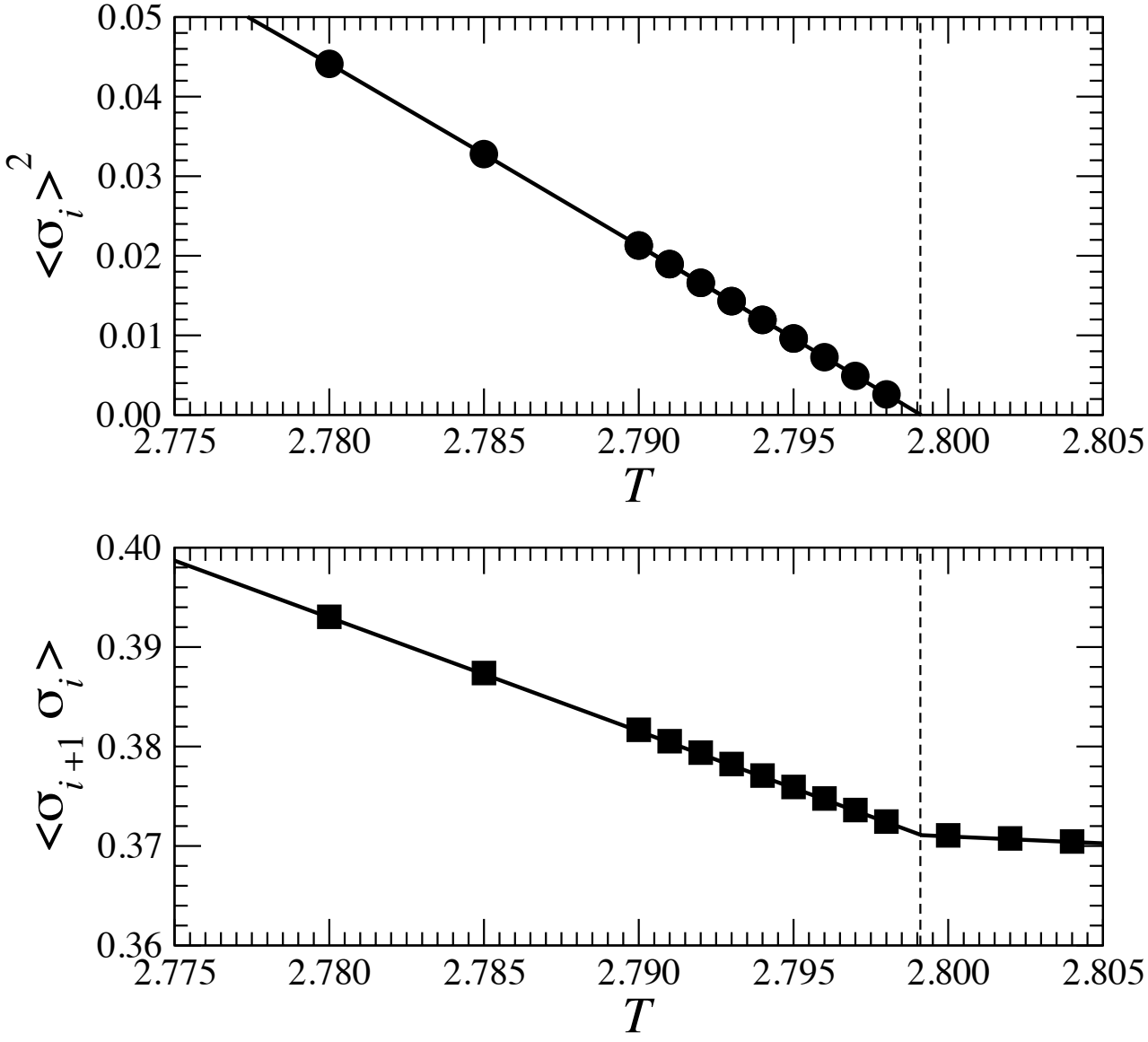






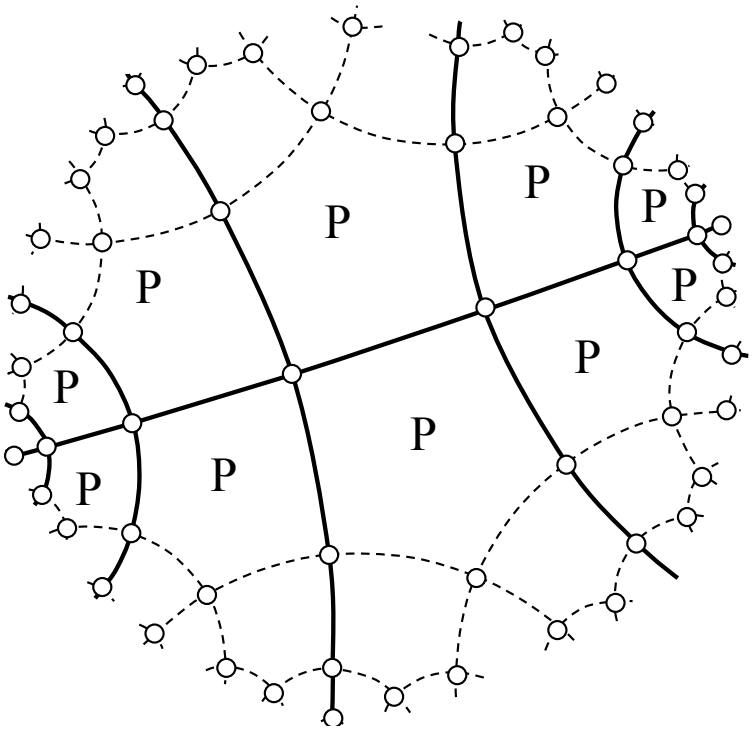


The Mean-Field Universality is observed.



... as it was conjectured by H. Shima and Y. Sakiniwa, J. Phys. A 39 (2006) 4921.

Transition is NOT Critical.



Column-to-Column Transfer Matrix P*P

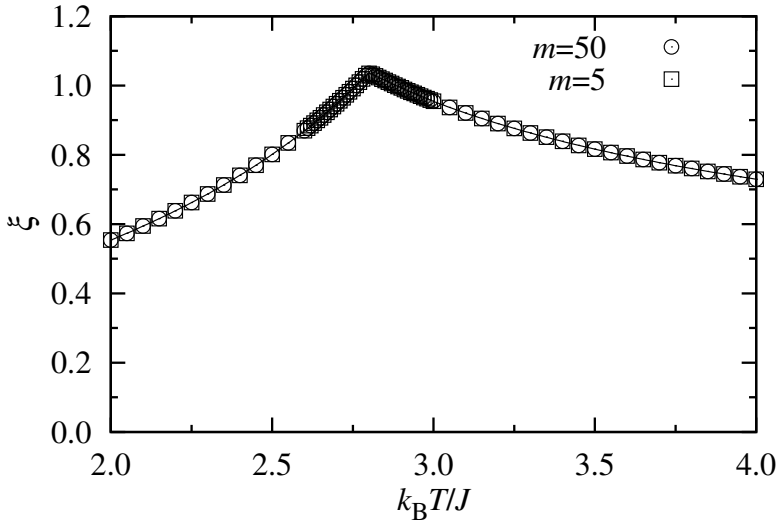


Fig. 2. Correlation Length ξ with respect to the temperature

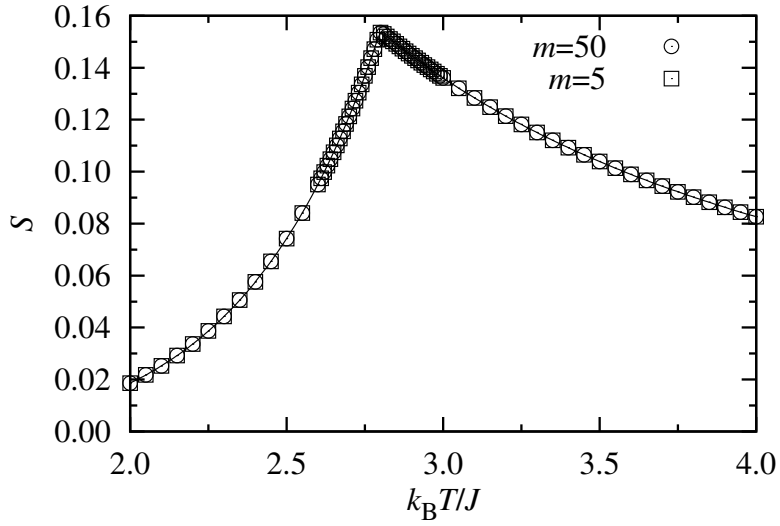
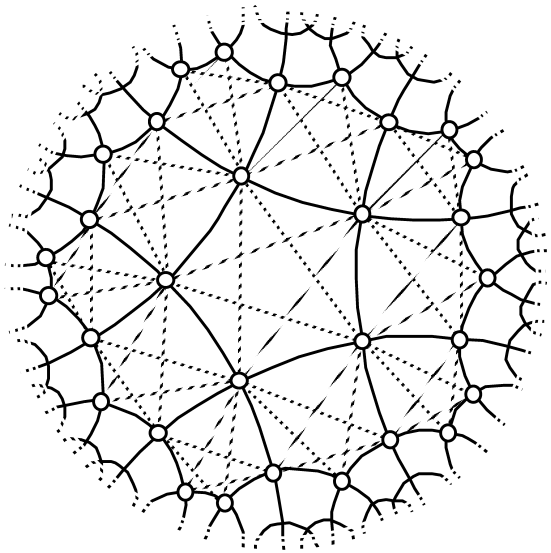
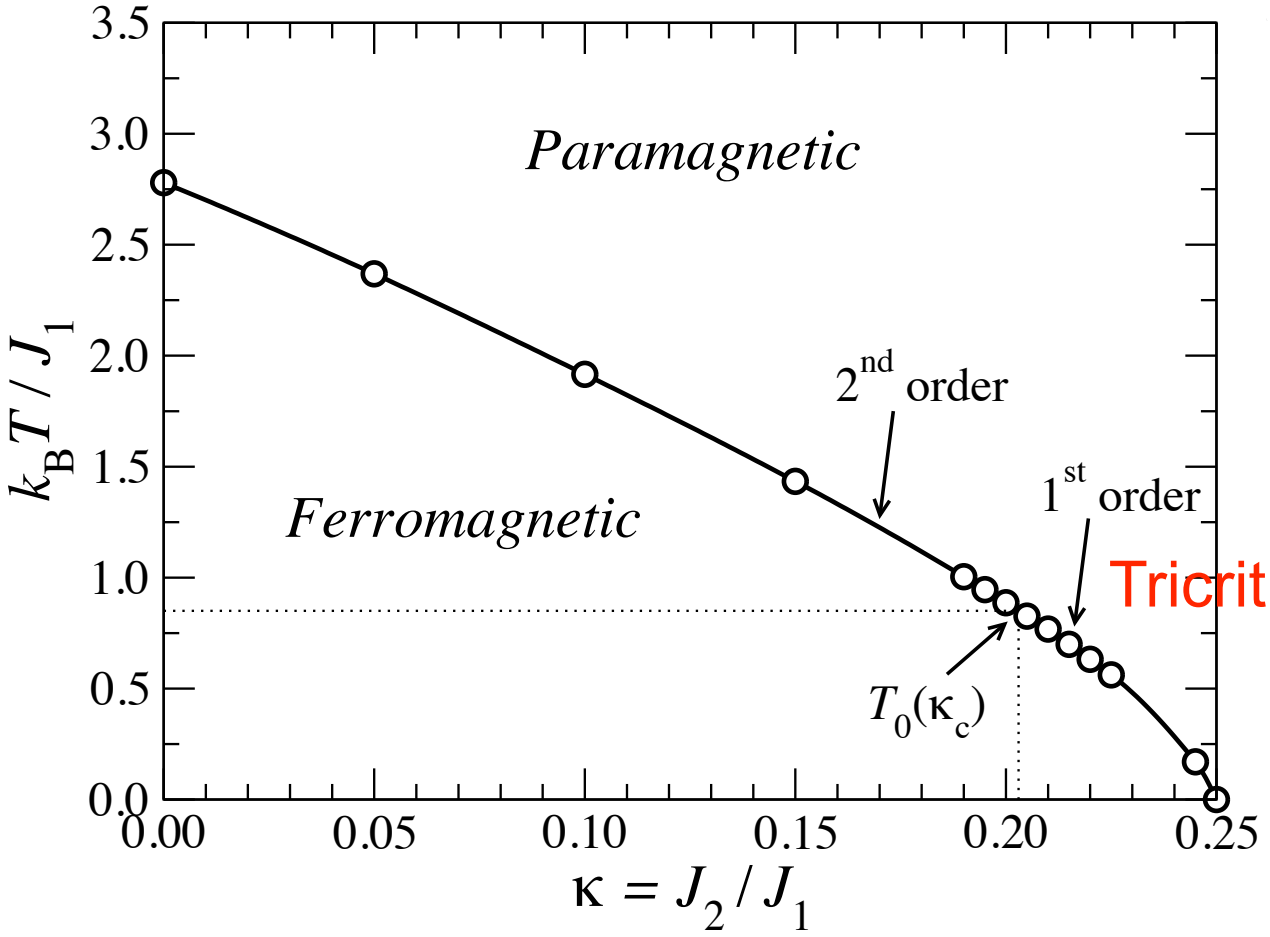


Fig. 3. Entanglement entropy of the MPS in Eq. (2.12).

NNN Ising model (J1-J2 Ising)

$$\mathcal{H} = -J_1 \sum_{\langle ij \rangle = \mathrm{NN}} \sigma_i \sigma_j + J_2 \sum_{\langle ik \rangle = \mathrm{NNN}} \sigma_i \sigma_k$$



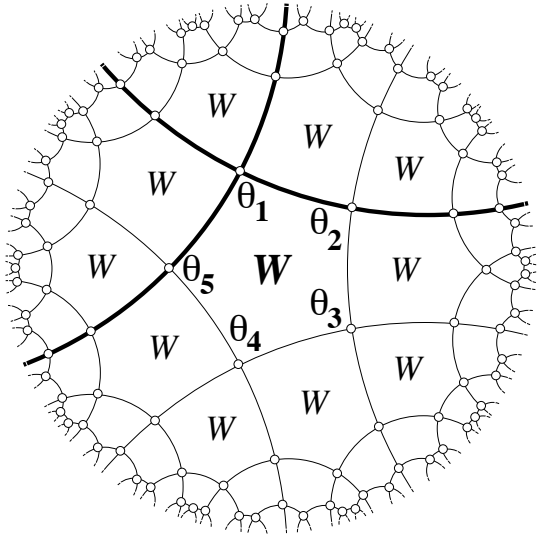


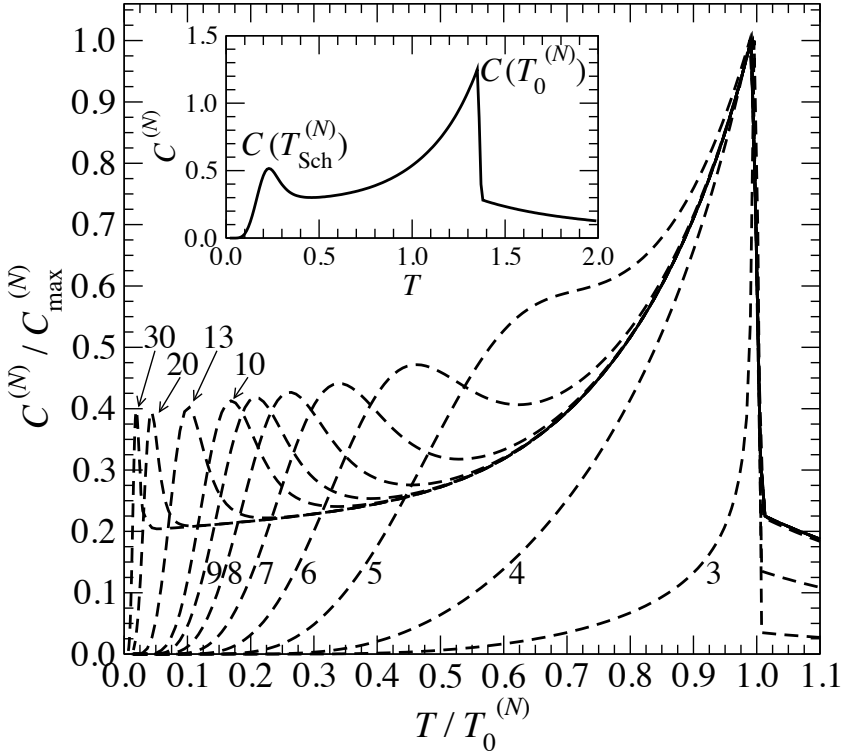
Tricritical (??) Point!

q-state Clock Models

$$W(\theta_1\,\theta_2\,\theta_3\,\theta_4\,\theta_5) = \prod_{i=1}^{5} \exp\left\{ \frac{J\cos\left(\theta_i - \theta_{i+1}\right)}{2\,k_{\rm B}T} \right\}$$

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \cos\left(\theta_i - \theta_j\right)$$





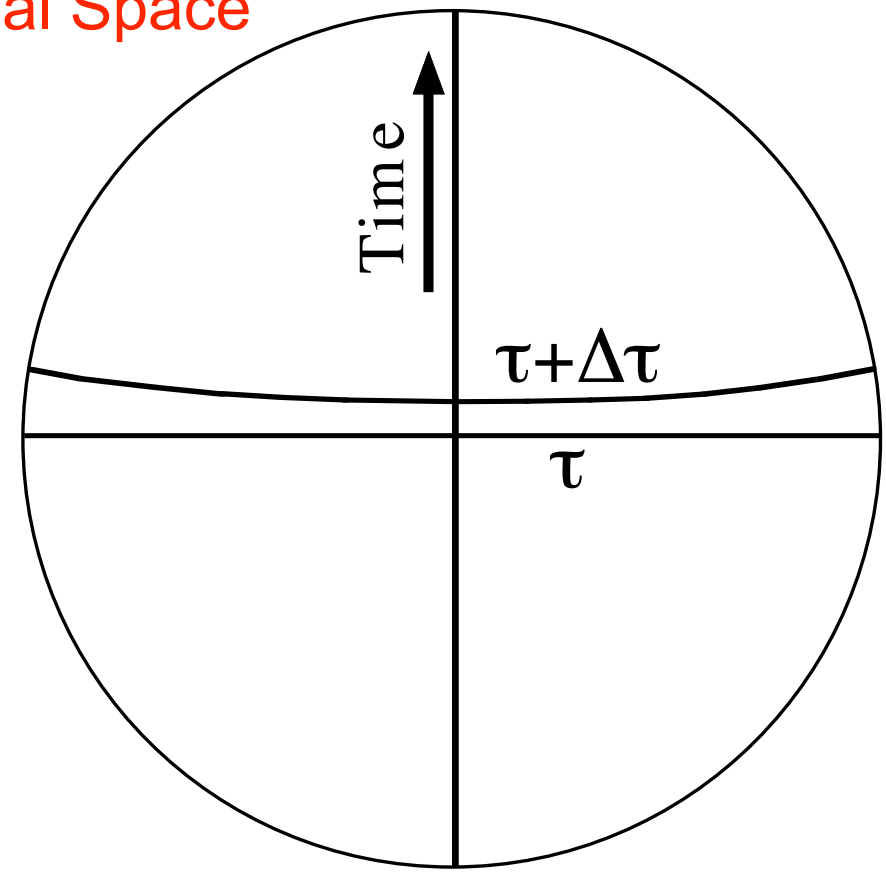
q=3: First Order Transition
q>4: Mean-Field 2nd Order

Toward Hyperbolic 1+1 dimensional Space

... if one discretize the space, one would obtain non-uniform Hamiltonian ...

$$H = \Sigma_j \cosh(j\lambda) \, h_{j,j+1}$$

Is the ground-state of such a Hamiltonian unifrom???



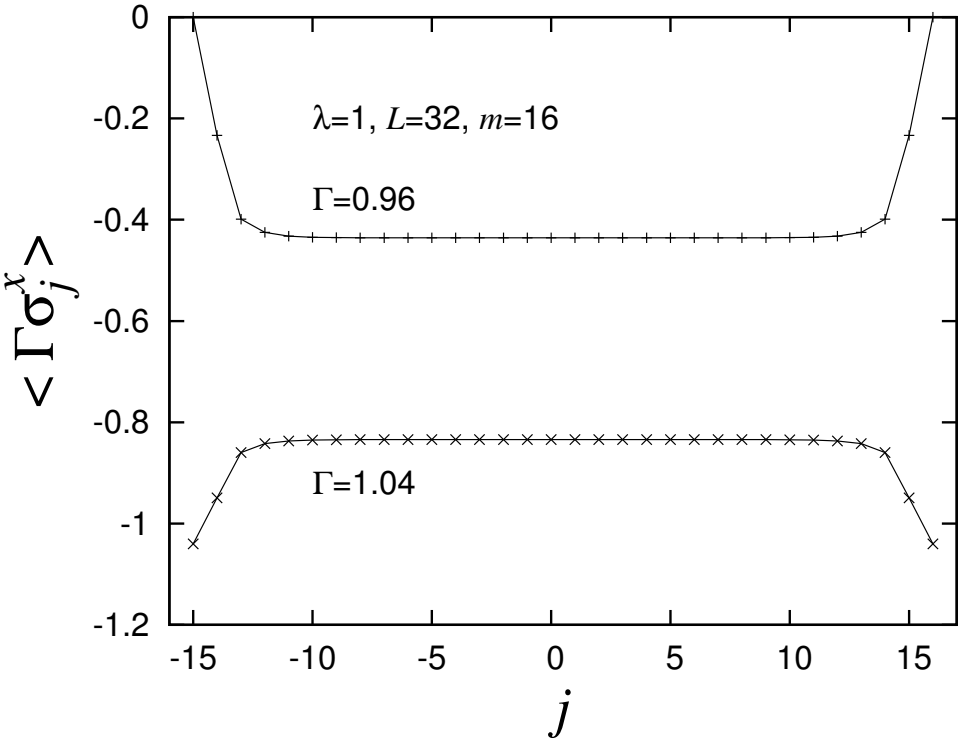
Poincare Disk

(*) one can consider the same sort of deformations on Sphere, Projective plane, Klein tube, etc.

Transverse Field Ising Model under Hyperbolic Deformation

$$H^{\mathrm{c}}(\lambda) \quad = \quad -J \sum_j \cosh\left[j\lambda\right] \sigma_j^z \sigma_{j+1}^z \quad -\Gamma \sum_j \cosh\left[\left(j-\tfrac{1}{2}\right)\lambda\right] \sigma_j^x$$

Deep inside the system, the ground state is quite uniform.



Correlation Length appears as the dumping rate of the boundary effect.

Correlation Length is inverse proportional to the deformation parameter.

Fig. 2. On-site transverse interaction $\langle \Gamma \sigma_j^x \rangle$.

Tips in DMRG:

Numerical Precision increases if one subtract the average of the Hamiltonian; this prevents the appearance of large matrix element in the block Hamiltonian. As a result, computation is stabilized.

We employ DMRG method[5–8] for the numerical determination of the ground state. A direct application of the finite-system DMRG algorithm encounters a numerical instability, which is caused by the blow-up of the energy scale $\cosh\left[j\lambda\right]$ with respect to $|j|$. In order to stabilize the computation, we treat

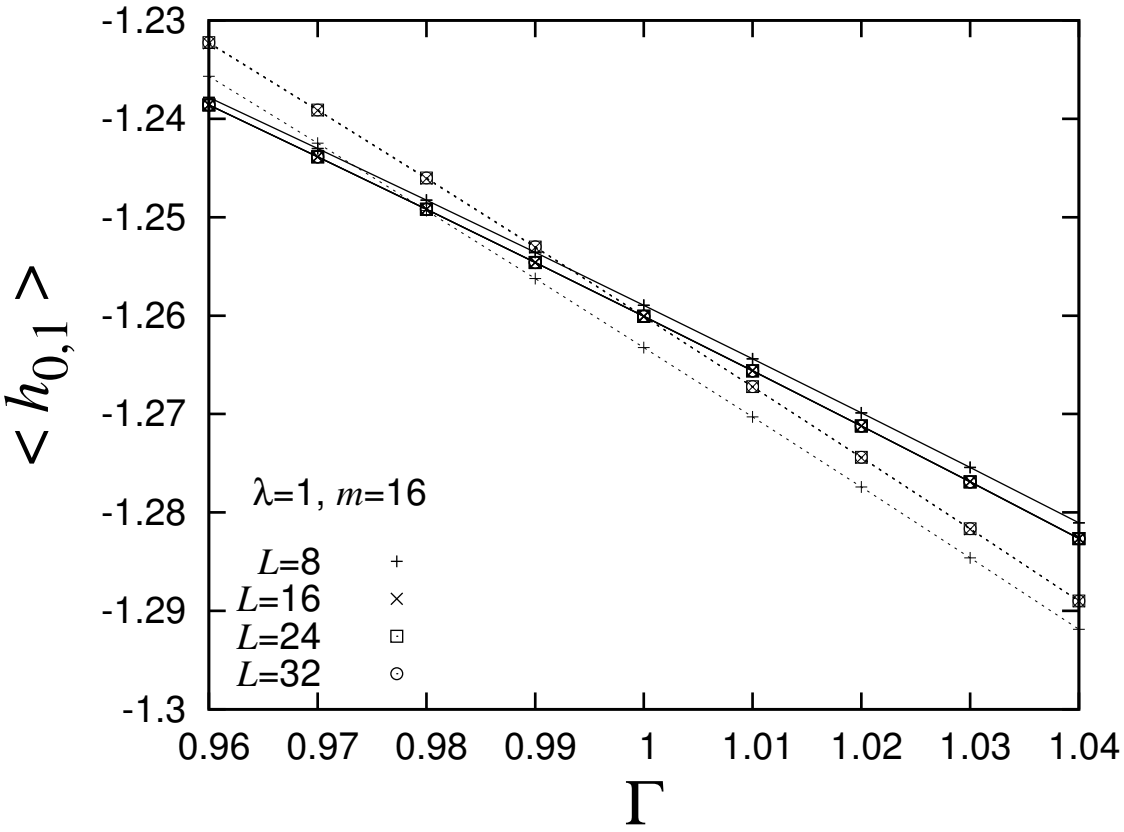
$$\begin{aligned} \tilde{H}^{\mathrm{c}}_{\mathrm{P};L}(\lambda) &\equiv H^{\mathrm{c}}_{\mathrm{P};L}(\lambda) - \left\langle H^{\mathrm{c}}_{\mathrm{P};L}(\lambda) \right\rangle \\ \tilde{H}^{\mathrm{c}}_{\mathrm{F};L}(\lambda) &\equiv H^{\mathrm{c}}_{\mathrm{F};L}(\lambda) - \left\langle H^{\mathrm{c}}_{\mathrm{F};L}(\lambda) \right\rangle \end{aligned} \tag{2.3}$$

instead of $H^{\mathrm{c}}_{\mathrm{P};L}(\lambda)$ and $H^{\mathrm{c}}_{\mathrm{F};L}(\lambda)$ directly, where $\langle\ \rangle$ denotes expectation value taken by the ground state. The smallest eigenvalue of both $\tilde{H}^{\mathrm{c}}_{\mathrm{P};L}(\lambda)$ and $\tilde{H}^{\mathrm{c}}_{\mathrm{F};L}(\lambda)$ is zero by definition. Since the ground state is not a priori

Imagine what happens if one repeat infinite system DMRG steps INFINITELY.

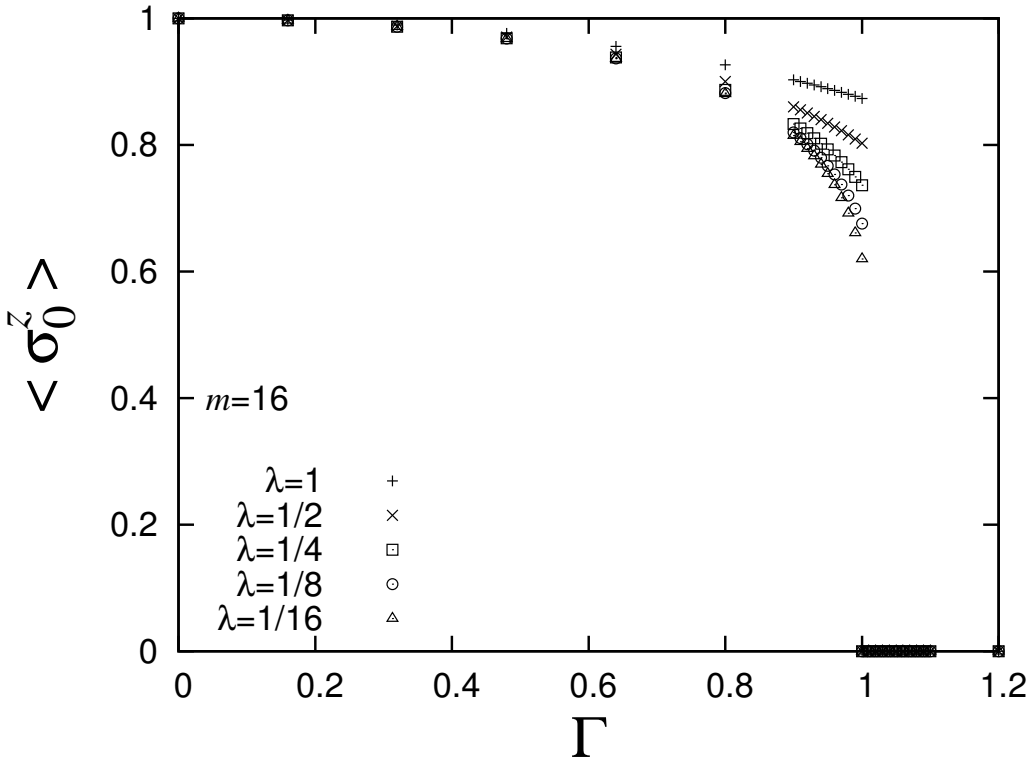
Energy CrossOver

Local Energy Density Operator: $h_{j,j+1} = -\left[\frac{\Gamma}{2}\sigma_j^x + J\sigma_j^z\sigma_{j+1}^z + \frac{\Gamma}{2}\sigma_{j+1}^x\right]$



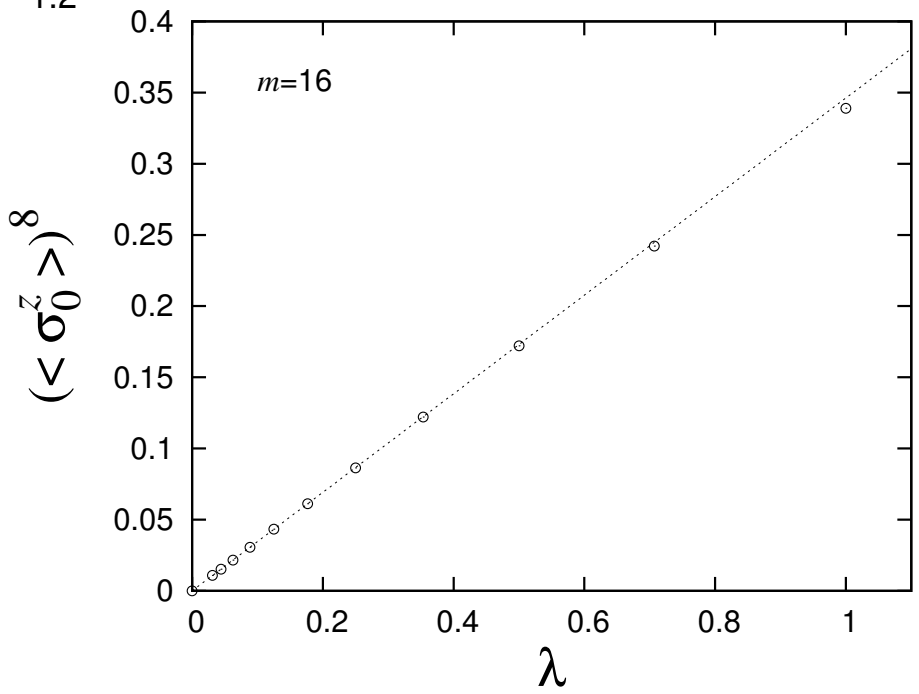
There is no singularity at the transition point gamma = 1.

>>> this is the First Order Phase Transition >>>



Spontaneous magnetization jumps at the transition point.

8–th power of the jump is proportional to the deformation parameter. The Ising Universality is detected, somehow.



Entanglement Entropy

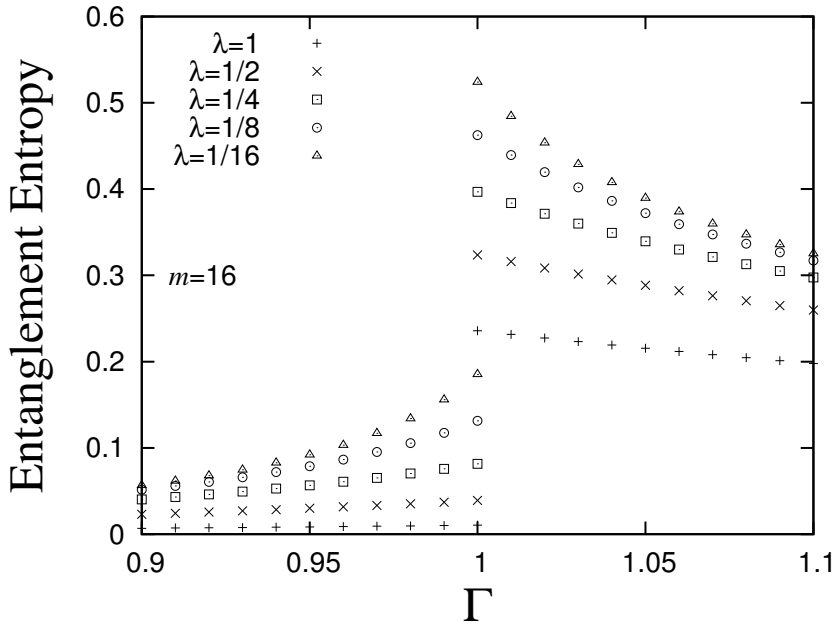
also jumps at the transition point.

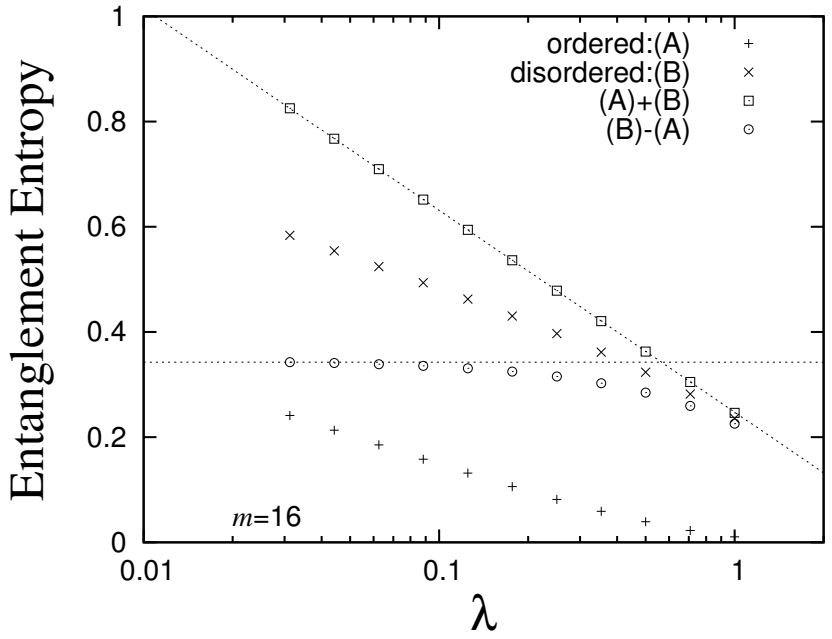
$$[(\mathrm{A})+(\mathrm{B})]/2 \text{ at } \Gamma = 1$$

$$\frac{1}{2}\cdot\frac{1}{6}\log\frac{1}{\lambda}+\frac{0.2475}{2}$$

$$(\mathrm{B})\text{-}(\mathrm{A}) \sim (1/2)\log 2$$

These are consistent with Ising Universality when the system size is restricted to 1/lambda..





Missing Link:

Classical (isotropic) Ising on the Hyperbolic Lattice shows 2nd order transition.

Quantum Ising under Hyperbolic Deformation shows 1st order transition.

One should construct the anisotropic limit.

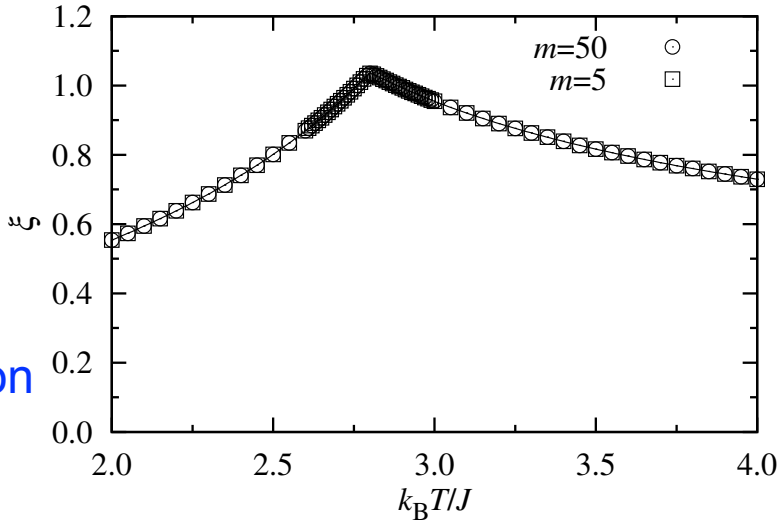
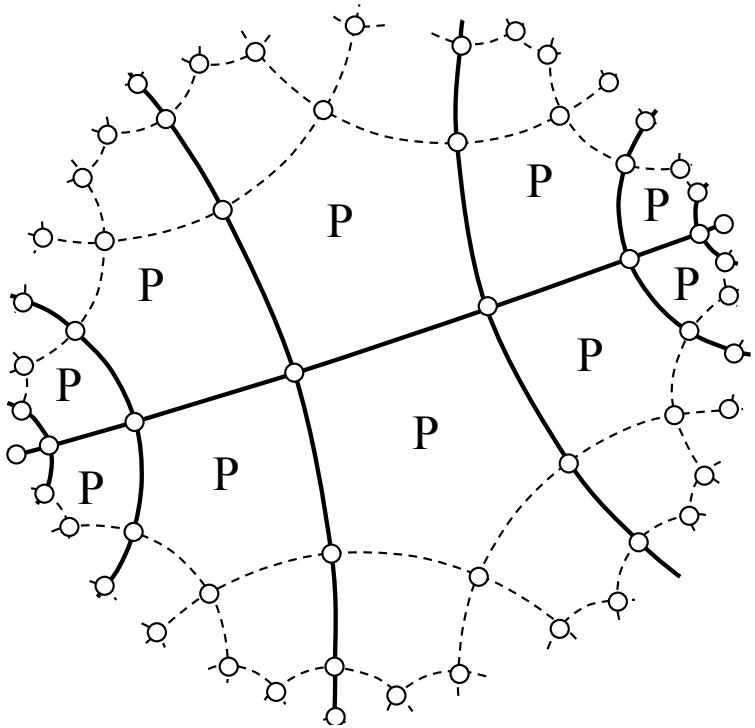


Fig. 2. Correlation Length ξ with respect to the temperature



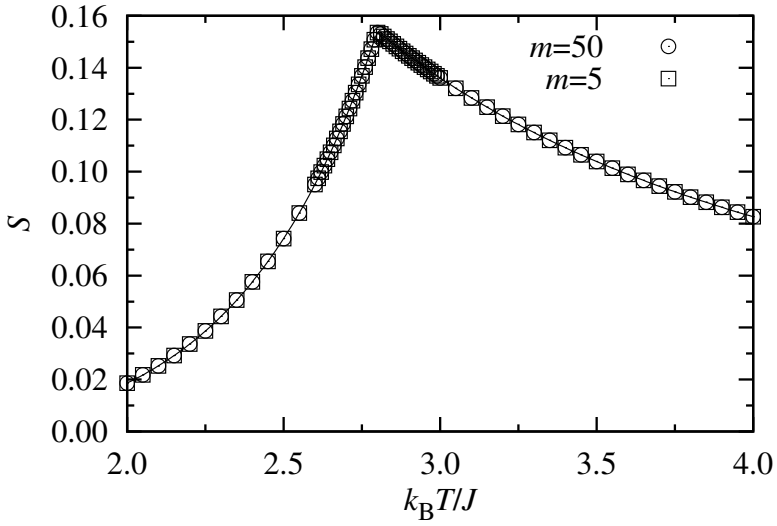
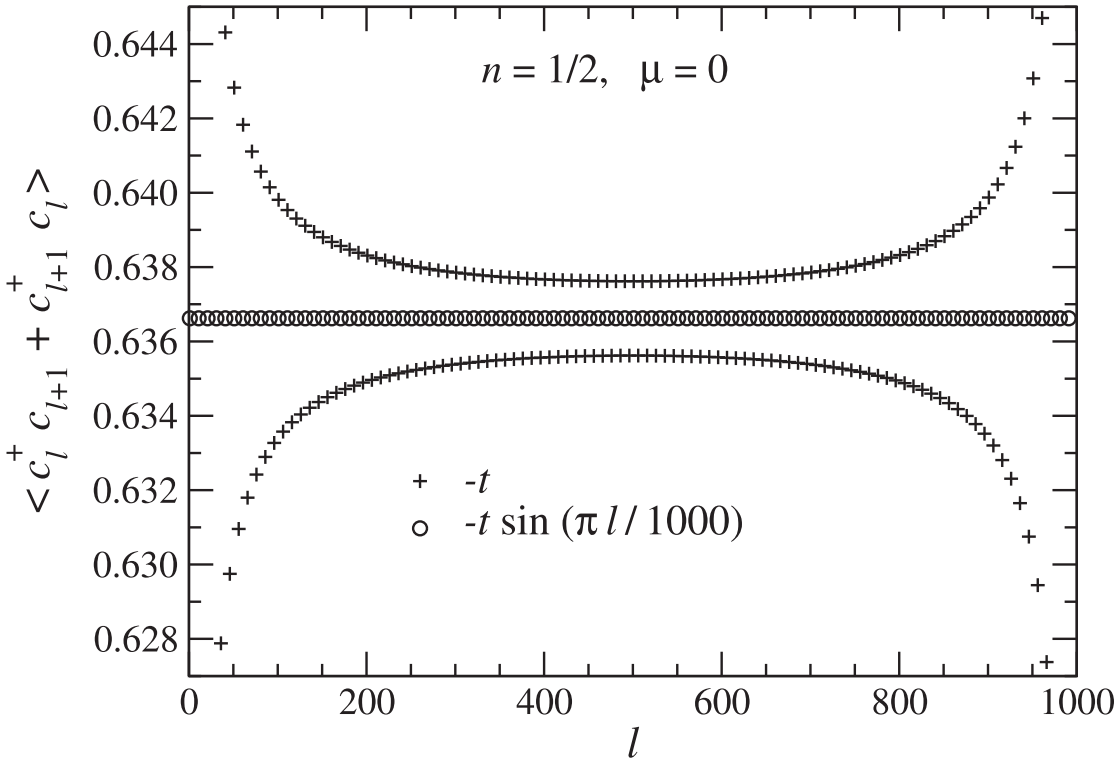


Fig. 3. Entanglement entropy of the MPS in Eq. (2.12).

A Generalization: Spherical Deformation

N-site tight binding Hamiltonian

$$\hat{H}_{\mathrm{S}} = -t \sum_{\ell=1}^{N-1} \sin^2 \frac{\ell\pi}{N} \left(\hat{c}_{\ell}^{\dagger} \hat{c}_{\ell+1} + \hat{c}_{\ell+1}^{\dagger} \hat{c}_{\ell} \right)$$



Boundary effect on the bond energy disappears completely!

A system under Open Boundary Condition gives data as efficient as those under Periodic Boundary Condition, under the spherical deformation.

◎ Conclusions

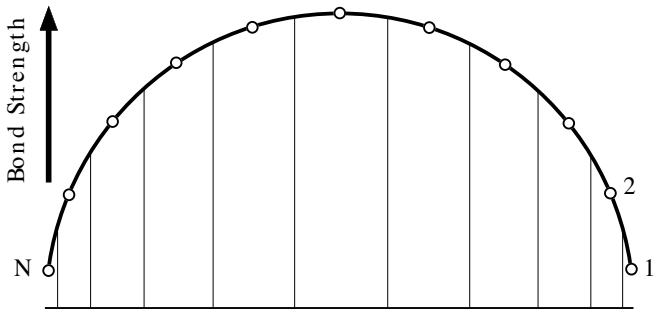
- Still there are many interest in 1D systems.
- Please memorize the term Hyperbolic Deformation and put it into Google.

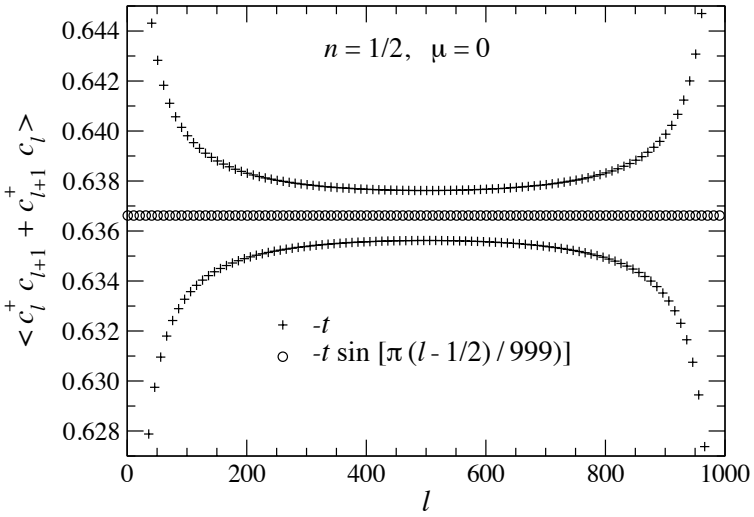
◎ Please visit the Web Pages

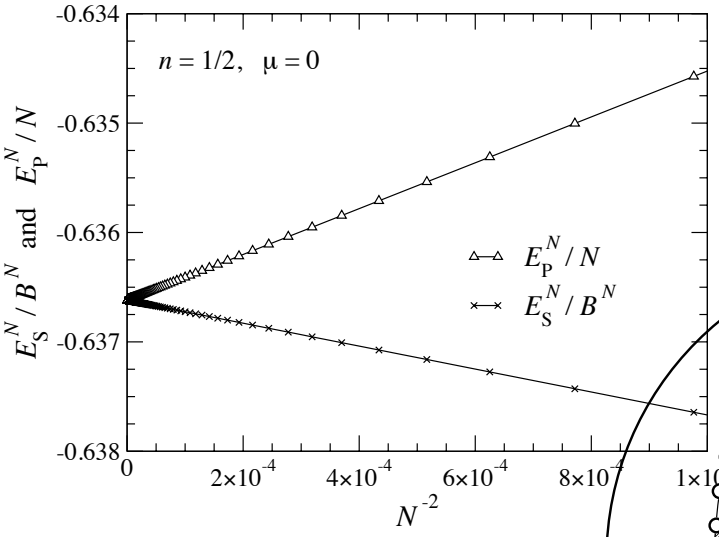
- http://quattro.phys.sci.kobe-u.ac.jp/nishino_e.html (My personal page)
- http://quattro.phys.sci.kobe-u.ac.jp/Hyperbolic/Hyperbolic.html (Hyperbolic)
- http://quattro.phys.sci.kobe-u.ac.jp/dmrg.html (List of arXiv preprints.)

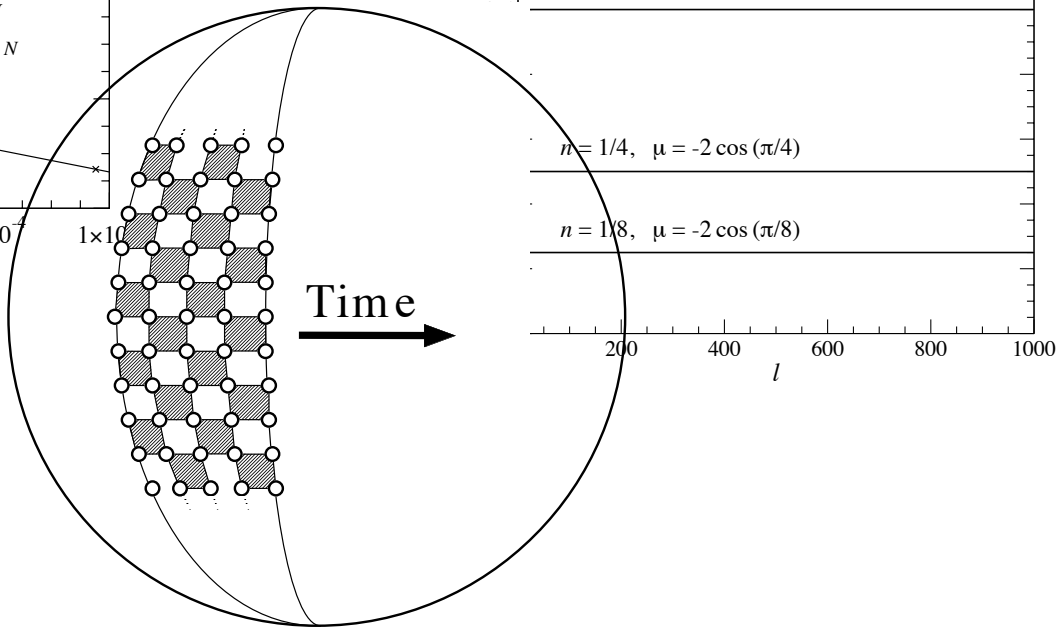
(**) I probably miss about 20 percent of articles each year. Recently xxx.lanl.gov provides key word search for the whole part of the submitted papers. Day by day I search the Key Word, DMRG, and put the missing articles to the list. For those years 1992-2003 I completed the task.

The book "Density-Matrix Renormalization" ed. I Peschel et al (Springer 1999) is NOT on line. Is there any way to make it open access?


Bond Strength
N
2
1


n = 1/2, μ = 0
-t
-t sin [π (l - 1/2) / 999)]
0.644
0.642
0.640
0.638
0.636
0.634
0.632
0.630
0.628
0
200
400
600
800
1000
l


n = 1/2, μ = 0
-0.634
-0.635
-0.636
-0.637
-0.638
0
2×10^-4
4×10^-4
6×10^-4
8×10^-4
1×10
N^-2


Time
n = 1/2, μ = 0
n = 1/4, μ = -2 cos (π/4)
n = 1/8, μ = -2 cos (π/8)
0.6
200
400
600
800
1000
l