Renormalization Group Approach to Casimir Force in Substance *

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1. Introduction

Casimir force , Casimir-Polder force , Van der Waals force ,
Forces caused by fluctuation due to microdynamics

- (nearly) free theory is involved. Main part is independent of the coupling(s)
- depends on the boundary parameters and the topology, macro property
- quantum effect (zero-point oscillation), micro property
- highly-delicate regularization is required, IR and UV, Summation formula

- Ambiguity of energy origin.

  Electro-Magnetic field in substance

$$\mathbf{D} = \varepsilon(\omega)\mathbf{E} \quad , \quad \mathbf{B} = \mu(\omega)\mathbf{H} \quad . \tag{1}$$

dielectric function (permittivity) ε, magnetic permeability μ: shows the substance property effectively. Can we approach it geometrically ?

2. Maxwell Equation in Substance

Electric and Magnetic Field

$$\text{Upper-index Fields} \ : \ \hat{\mathbf{D}}(t,\mathbf{x}) = (\hat{D}^i(x)) \quad , \quad \hat{\mathbf{B}}(t,\mathbf{x}) = (\hat{B}^i(x)),$$

$$\text{Lower-index Fields} \ : \ \hat{\mathbf{E}}(t,\mathbf{x}) = (\hat{E}_i(x)) \quad , \quad \hat{\mathbf{H}}(t,\mathbf{x}) = (\hat{H}_i(x)),$$

$$i = 1,2,3 \quad , \quad \mathbf{x} = (x,y,z) \quad , \quad (x^\mu) = (t,\mathbf{x}) \quad , \quad \mu = 0,1,2,3 \tag{2}$$

Dielectric function, Magnetic permeability (general form)

$$\hat{D}^i(x) = \hat{\varepsilon}^{ij}(x)\hat{E}_j(x) \quad , \quad \hat{B}^i(x) = \hat{\mu}^{ij}(x)\hat{H}_j(x) \quad . \tag{3}$$

$$\text{div}\hat{\mathbf{D}} = \partial_i \hat{D}^i = 0 \quad \text{electric charge density} = 0$$

$$\text{div}\hat{\mathbf{B}} = \partial_i \hat{B}^i = 0 \quad \text{magnetic charge density} = 0 \quad . \tag{4}$$

Ampére's Law

$$\partial_t \hat{D}^i - \epsilon^{ijk} \partial_j \hat{H}_k = 0 \quad \text{or} \quad \partial_t \hat{\mathbf{D}} - \nabla \times \hat{\mathbf{H}} = 0$$

$$(\text{electric current density} = 0) \tag{5}$$

Faraday's Law

$$\partial_t \hat{B}^i + \epsilon^{ijk} \partial_j \hat{E}_k = 0 \quad \text{or} \quad \partial_t \hat{\mathbf{B}} + \nabla \times \hat{\mathbf{E}} = 0 \quad . \tag{6}$$

Faraday's law is solved by the vector and scalar potentials.

$$\hat{\mathbf{E}} \quad , \quad \hat{\mathbf{B}} \quad \rightarrow \quad \hat{\mathbf{A}} \quad , \quad \hat{\phi}$$

$$\hat{E}_i(\mathbf{x},t) = -\partial_t \hat{A}_i(\mathbf{x},t) - \partial_i \hat{\phi}(\mathbf{x},t)$$
$$\hat{B}^i(\mathbf{x},t) = \epsilon^{ijk}\partial_j \hat{A}_k(\mathbf{x},t) \quad . \tag{7}$$

time t $\quad to \quad$ frequency ω (Fourier expansion)

$$\hat{\mathbf{D}}(t,\mathbf{x}) = \int_{-\infty}^{\infty} \mathbf{D}(\omega,\mathbf{x})\mathrm{e}^{i\omega t} d\omega \quad , \quad \hat{\mathbf{E}}(t,\mathbf{x}) = \int_{-\infty}^{\infty} \mathbf{E}(\omega,\mathbf{x})\mathrm{e}^{i\omega t} d\omega$$
$$\hat{\mathbf{B}}(t,\mathbf{x}) = \int_{-\infty}^{\infty} \mathbf{B}(\omega,\mathbf{x})\mathrm{e}^{i\omega t} d\omega \quad , \quad \hat{\mathbf{H}}(t,\mathbf{x}) = \int_{-\infty}^{\infty} \mathbf{H}(\omega,\mathbf{x})\mathrm{e}^{i\omega t} d\omega$$
$$. \tag{8}$$

$$D^i(\omega,\mathbf{x}) = \varepsilon^{ij}(\omega)E_j(\omega,\mathbf{x}) \quad , \quad B^i(\omega,\mathbf{x}) = \mu^{ij}(\omega)H_j(\omega,\mathbf{x}) \quad . \tag{9}$$

$$t - \text{representation} \quad \rightarrow \quad \omega - \text{representation}$$

$$\hat{\mathbf{A}}(t,\mathbf{x}) = \int_{-\infty}^{\infty} \mathbf{A}(\omega,\mathbf{x})\mathrm{e}^{i\omega t} d\omega \quad , \quad \hat{\phi}(t,\mathbf{x}) = \int_{-\infty}^{\infty} \phi(\omega,\mathbf{x})\mathrm{e}^{i\omega t} d\omega \quad ,$$

$$E_i(\omega,\mathbf{x}) = -i\omega A_i(\omega,\mathbf{x}) - \partial_i \phi(\omega,\mathbf{x}) \quad \text{or} \quad \mathbf{E}(\omega,\mathbf{x}) = -i\omega \mathbf{A}(\omega,\mathbf{x}) - \nabla\phi(\omega,\mathbf{x})$$

$$B^i(\omega,\mathbf{x}) = \epsilon^{ijk}\partial_j A_k(\omega,\mathbf{x}) \quad \text{or} \quad \mathbf{B}(\omega,\mathbf{x}) = \nabla \times \mathbf{A}(\omega,\mathbf{x}) \quad . \tag{10}$$

$\mathbf{E}(\omega,\mathbf{x})$ and $\mathbf{B}(\omega,\mathbf{x})$ are unchanged under the gauge transformation.

$$A_i \quad \rightarrow \quad A_i + \partial_i \Lambda \quad , \quad \phi \quad \rightarrow \quad \phi - i\omega\Lambda \quad \text{where } \Lambda = \Lambda(\omega,\mathbf{x}) \quad . \tag{11}$$

For simplicity, we consider the diagonal case.

$$\varepsilon^{ij} = \varepsilon(\omega)\delta^{ij} \quad , \quad (\mu^{-1})_{ij} = \mu^{-1}(\omega)\delta_{ij} \quad . \tag{12}$$

Gauge 1. $\partial_i\{i\omega\phi + (\varepsilon\mu)^{-1}\text{div}\mathbf{A}\} = 0$

Ampére's law gives the field eq. of $\mathbf{A}$

$$(\Delta + \omega^2\varepsilon\mu)\mathbf{A}(\omega, \mathbf{x}) = 0 \quad , \quad \mathbf{E} = -i\omega\mathbf{A} - \frac{i}{\omega\varepsilon\mu}\nabla(\text{div}\mathbf{A}) \quad . \tag{13}$$

Note: When ε and μ are constants, $\hat{\mathbf{A}}(t, \mathbf{x})$ satisfies the free wave equation with velocity of light $v = \frac{1}{\sqrt{\varepsilon\mu}}$.

Gauge 2. $\partial_i\phi = 0$ (Landau-Lifshitz textbook)

The fields eq. of $\mathbf{A}$

$$\begin{aligned} \Delta\mathbf{A} - \nabla(\text{div}\mathbf{A}) + \omega^2\varepsilon\mu\mathbf{A} = 0 \quad , \\ \mathbf{E} = -i\omega\mathbf{A} \quad . \end{aligned} \tag{14}$$

3. Geometry in (k^μ)=(ω, k^i) Space

The energy of the substance is given by *** geo1

$$\int d^3x \int d\omega \; \mathcal{E}$$

$$= \int d^3x \int d\omega \frac{1}{2}(\varepsilon^{ij} E_i E_j + \mu^{-1}{}_{ij} B^i B^j) = \int d^3x \int d\omega \frac{1}{2}\mu^{-1} \mathbf{A} \cdot (\Delta + \omega^2 \varepsilon\mu)\mathbf{A} \quad . \tag{15}$$

We propose *** geo1b

$$(\mu^{-1})_{ij} = \sqrt{g(\omega)} g_{ij}(\omega) \quad , \quad \varepsilon^{ij}(\omega) = \sqrt{g(\omega)} g^{ij}(\omega) \quad , \tag{16}$$

where $g_{ij}(\omega)$ is some metric introduced next, and $g = \det g_{ij}$.

Let us consider three typical metrics in (ω, K^i) space. *** geo2

$$1.\ \text{Minkowski} \quad : \quad ds^2 = -d\omega^2 + \sum_{i=1}^{3} dK^{i^2}$$

$$2.\ \text{dS}_4 \quad : \quad ds^2 = -d\omega^2 + \mathrm{e}^{2H\omega} \sum_{i=1}^{3} dK^{i^2}$$

$$3.\ \text{AdS}_4 \quad : \quad ds^2 = (dK^3)^2 + \mathrm{e}^{-2H|K^3|}(-d\omega^2 + (dK^1)^2 + (dK^2)^2) \tag{17}$$

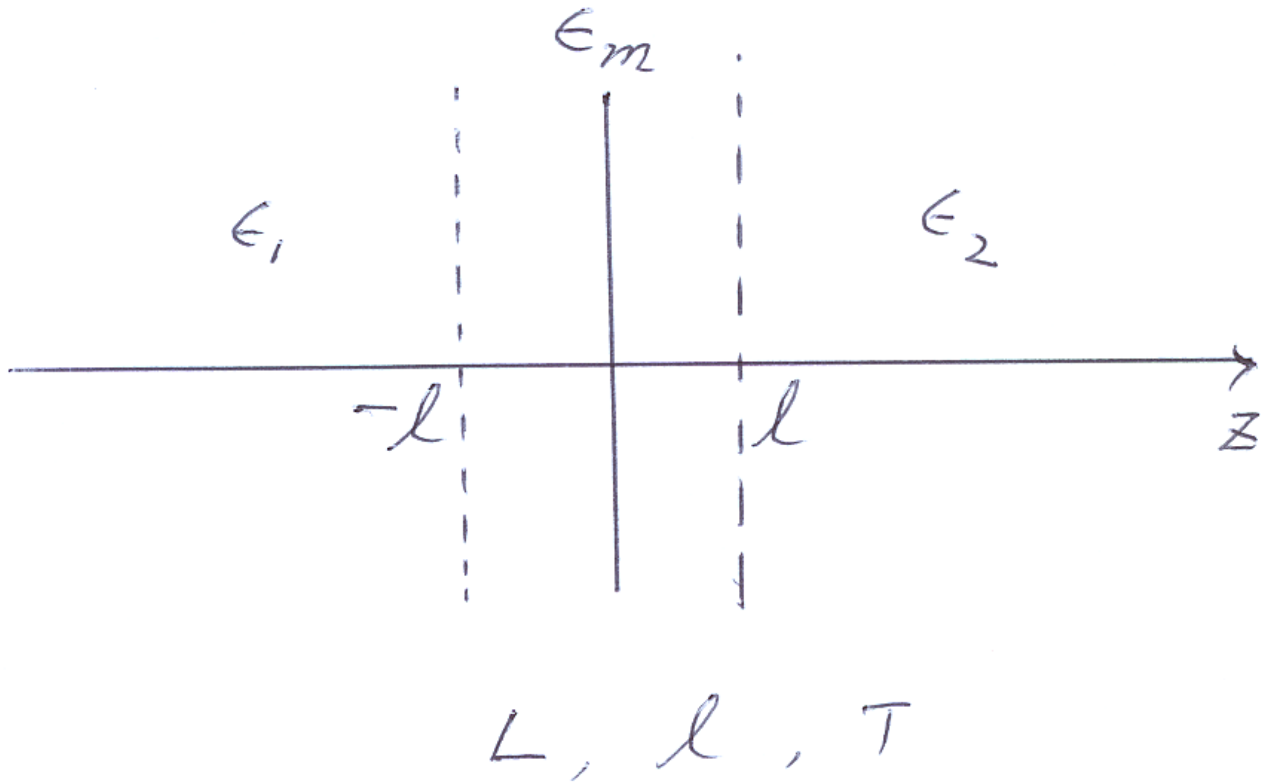
To specify (parametrize) 3D metric $g_{ij}(\omega)$, introduce 3D *hypersurface*. *** geo3

$$\text{On-shell Condition} \quad : \quad (K^i)^2 \ = \ r(\omega)^2 \quad , \tag{18}$$

The induced metric g_{ij} is given by *** geo4

$$g_{ij}(\omega) = \delta_{ij} - \frac{k^i k^j}{(r\dot{r})^2} \quad . \tag{19}$$

Figure 1: Configuration of Substance $\varepsilon_1, \varepsilon_m, \varepsilon_2$.



4. Lifshitz Theory 1956

Free energy formula for the $\varepsilon_1, \varepsilon_m, \varepsilon_2$ $(\mu = 1)$ substance of the structure: Fig.1 .

Simplified model of the previous Maxwell theory Kenneth and Klich '06

$$S = \frac{1}{2}\int d^3x \int \frac{d\omega}{2\pi}\phi^*_\omega(\Delta + \omega^2\varepsilon(\omega))\phi_\omega \quad , \quad \phi^*_\omega = \phi_{-\omega} \quad , \quad \varepsilon(\omega) = 1 + \chi(\omega) \quad .(20)$$

When $\varepsilon(\omega) = \varepsilon_1$(const.), above is 3+1 massless complex free scalar.

$$S = \frac{1}{2}\int d^3x \int \frac{dt}{2\pi}\hat{\phi}^*(\mathbf{x},t)(\Delta - \varepsilon_1\frac{\partial^2}{\partial t^2})\hat{\phi}(\mathbf{x},t) \quad , \quad \hat{\phi}(\mathbf{x},t) = \int_{-\infty}^{\infty}\phi_\omega(\mathbf{x})\mathrm{e}^{i\omega t}d\omega \quad .(21)$$

The field equation

$$(\Delta + \omega^2 \varepsilon(\omega))\phi_\omega = 0 \quad , \quad \phi_\omega(\mathbf{x}_\perp, z) = \tilde{\phi}_\omega(z) \mathrm{e}^{i\mathbf{q}\cdot\mathbf{x}_\perp} \quad . \tag{22}$$

This system is in the thermal equilibrium at temperature T.

$$\begin{aligned} \text{Periodicity} \quad : \quad t \quad \rightarrow \quad t + \frac{1}{T} \quad , \\ \omega_n = \frac{2\pi T}{\hbar} n \quad . \end{aligned} \tag{23}$$

In the plane perpendicular to z-axis,

$$\begin{aligned} \text{Periodicity} \quad : \quad \mathbf{x}_\perp = (x, y) \quad \rightarrow \quad (x + L, y + L) \quad , \\ \mathbf{q}_{(n_x, n_y)} = (\frac{2\pi}{L} n_x, \frac{2\pi}{L} n_y) \quad . \end{aligned} \tag{24}$$

$$(-\mathbf{q}^2 + {\partial_z}^2 + \omega^2\varepsilon(\omega))\tilde{\phi}_\omega(z) = 0 \quad . \tag{25}$$

For z-dependence, take

$$\tilde{\phi}_\omega(z) = A(\omega)\mathrm{e}^{\rho z} + B(\omega)\mathrm{e}^{-\rho z} \quad , \quad -\mathbf{q}^2 + {\rho_\alpha}^2 + \omega^2\varepsilon_\alpha(\omega) = 0 \quad (\alpha = 1, m, 2) \quad . \tag{26}$$

Wave function for each region

$$\begin{aligned} z < -l \qquad & \tilde{\phi}_\omega(z) = A(\omega)\mathrm{e}^{\rho_1 z} \quad , \quad \text{region 1} \\ -l < z < l \qquad & \tilde{\phi}_\omega(z) = C_1(\omega)\mathrm{e}^{\rho_m z} + C_2(\omega)\mathrm{e}^{-\rho_m z} \quad , \quad \text{region m} \\ z > l \qquad & \tilde{\phi}_\omega(z) = B(\omega)\mathrm{e}^{-\rho_2 z} \quad \text{region 2} \end{aligned} \tag{27}$$

$$\Delta = 1 - \frac{(\rho_1 - \rho_m)(\rho_2 - \rho_m)}{(\rho_1 + \rho_m)(\rho_2 + \rho_m)}\mathrm{e}^{-4\rho_m l} = 0 \quad . \tag{28}$$

Add periodicity to z-direction.

$$\text{Periodicity} \quad : \quad z \quad \rightarrow \quad z + L \quad ,$$
$$\mathbf{q}_{(n_x,n_y,n_z)} = (\frac{2\pi}{L}n_x, \frac{2\pi}{L}n_y, \frac{2\pi}{L}n_z) \quad . \tag{29}$$

$$\mathrm{e}^{-F_\chi} = \int \mathcal{D}\phi_\omega \mathcal{D}\phi^*_\omega \mathrm{e}^{iS[\phi^*,\phi;\chi_1,\chi_m]}$$
$$= \det(\Delta + \omega^2\epsilon_\alpha(\omega)) = \exp \mathrm{Tr} \ln \left(\Delta + \omega^2(1 + \chi_\alpha(\omega))\right) \quad , \tag{30}$$

$\epsilon_\alpha = 1 + \chi_\alpha$ is given by

$$\text{In } \mathrm{R}_1 \quad 1 + \chi_1(\omega) = \frac{1}{\omega^2}(\frac{2\pi}{L})^2(n_x^2 + n_y^2 + n_z^2) \quad ,$$

$$\text{In } R_m \quad 1+\chi_m(\omega) = \frac{1}{\omega^2}(\frac{2\pi}{L})^2(n_x^2+n_y^2+n_z^2) \quad , \tag{31}$$

REGULARIZATION 1

$$F_C \equiv F_\chi - F_{\chi=0} = -\mathrm{Tr}\ln(1+\frac{\omega^2\chi_\alpha(\omega)}{\Delta+\omega^2}) \quad . \tag{32}$$

REGULARIZATION 2 (Entanglement)

$$F \equiv F_C(R_1 \cup R_m) - R_C(R_1) - R_C(R_m) \quad . \tag{33}$$

Finally, well-defined (finite) quantity Kenneth and Klich 2008

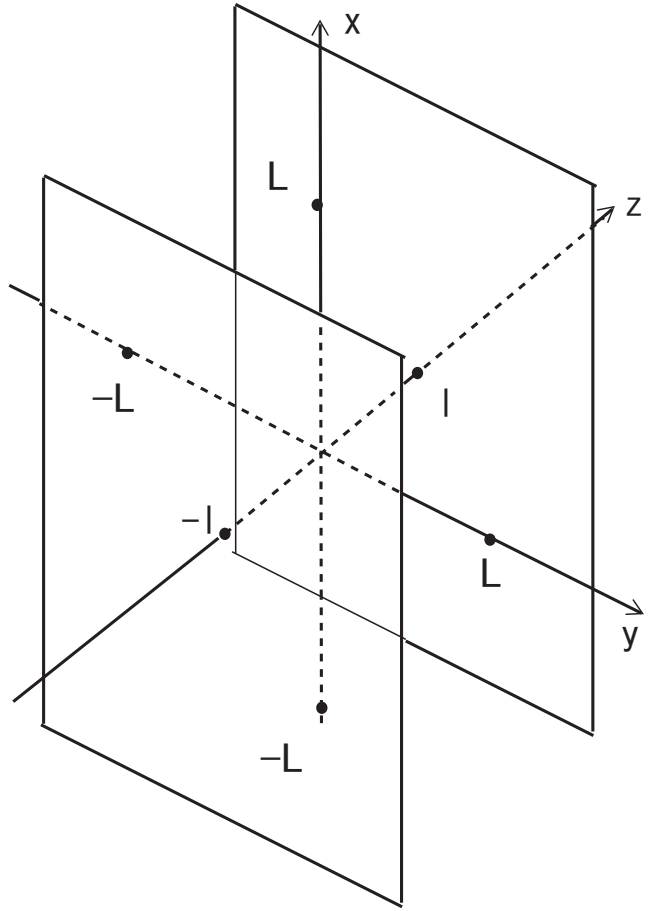
$$F = -\mathrm{Tr}\ln(1 - T_1 G_{1m} T_m G_{m1}) \quad ,$$

$$T_1 = \frac{\omega^2}{1+\omega^2\chi_1 G_{11}} \quad , \quad T_m = \frac{\omega^2}{1+\omega^2\chi_m G_{mm}} \quad . \tag{34}$$

G and $G_{\alpha\beta}$ is free propagators:

$$G = \begin{pmatrix} G_{11} & G_{1m} \\ G_{m1} & G_{mm} \end{pmatrix} \quad , \quad G(\mathbf{x},\mathbf{x}') =< \mathbf{x}|\frac{1}{\Delta+\omega^2}|\mathbf{x}' > \quad ,$$
$$(\Delta+\omega^2)G(\mathbf{x},\mathbf{x}') =< \mathbf{x}|\mathbf{x}' >= \delta(\mathbf{x}-\mathbf{x}') \quad , \quad \mathbf{x},\mathbf{x}' \in R_1 \cup R_m \quad . \tag{35}$$

Figure 2: Configuration of Casimir energy measurement.



5. Ordinary Regularization for Casimir Energy

1+3 D electromagnetism (free field theory) in Minkwski space:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \quad . \tag{36}$$

2 perfectly-conducting plates parallel with the separation $2l$ in the x-direction. As for y- and z-directions,the periodicity $2L$ for the IR regularization.

$$\text{Periodicity} \ : \quad x \to x + 2l \quad , \quad y \to y + 2L \quad , \quad z \to z + 2L \quad , \qquad L \gg l \quad , \tag{37}$$

the eigen frequencies and Casimir energy are

$$\omega_{n,m_y,m_z} = \sqrt{(n\frac{\pi}{l})^2 + (m_y\frac{\pi}{L})^2 + (m_z\frac{\pi}{L})^2} \quad ,$$
$$E_{Cas} = 2 \cdot \sum_{n,m_y,m_z \in \mathbf{Z}} \frac{1}{2}\omega_{n,m_y,m_z} \quad \geq 0 \quad , \tag{38}$$

($\mathbf{Z}$: all integers) $\frac{1}{2}\omega_{n,m_y,m_z}$ is the zero-point oscillation energy. Introducing the cut-off function $g(x)$ ($= 1$ for $0 < x < 1$, 0 for otherwise),

$$E^{\Lambda}_{Cas} = \sum_{n,m_y,m_z \in \mathbf{Z}} \omega_{n,m_y,m_z} g\left(\frac{\omega_{n,m_y,m_z}}{\Lambda}\right) \quad \geq 0 \quad , \quad \Lambda \ : \ \text{UV-CutOff} \ . \tag{39}$$

take the continuum limit $L \to \infty,\ L \ll l \to \infty$.

$$E_{Cas}^{\Lambda 0} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dk_y dk_z}{(\frac{\pi}{L})^2} \int_{-\infty}^{\infty} \frac{dk_x}{\frac{\pi}{l}} \sqrt{k_x^2 + k_y^2 + k_z^2}\ g(\frac{k}{\Lambda})$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty|k|\leq\Lambda}^{\infty} \frac{dk_x dk_y dk_z}{(\frac{\pi}{L})^2 \frac{\pi}{l}} \sqrt{k_x^2 + k_y^2 + k_z^2} \quad \geq 0 \quad . \tag{40}$$

Note that E_{Cas}, E_{Cas}^{Λ} and $E_{Cas}^{\Lambda 0}$ are all positive-definite. In a familiar way, regarding $E_{Cas}^{\Lambda 0}$ as the origin of the energy scale, we consider the quantity $u = (E_{Cas}^{\Lambda} - E_{Cas}^{\Lambda 0})/(2L)^2$ as the physical Casimir energy and evaluate it with the help of the Euler-MacLaurin formula as $u = (\pi^2/(2l)^3)\ (B_4/4!) = -(\pi^2/720)(1/(2l)^3) < 0$. The final result is negative. In the present analysis we take a new regularization which keeps positive-definiteness.

6. New Regularization for Casimir Energy

First we re-express $E^{\Lambda 0}_{Cas}$ using a simple identity : $l = \int_0^l dw$ (w: a regularization axis).

$$E^{\Lambda 0}_{Cas}/(2L)^2 = \frac{1}{2^2\pi^3}\int_0^l dw \int_{k\leq\Lambda} P(k) 2\pi k^2 dk$$

$$= \frac{1}{2^2\pi^3}\int_0^l dw(-1)\int_{r\geq\Lambda^{-1}} P(1/r)(-1)2\pi r^{-4} dr \quad .$$

$$P(k) \equiv k \quad , \quad r \equiv \frac{1}{k} \quad , \tag{41}$$

where the integration variable changes from the momentum (k) to the coordinate

($r = \sqrt{x^2 + y^2 + z^2}$). The integration region in (R, w)-space is the infinite rectangular shown in Fig.3.

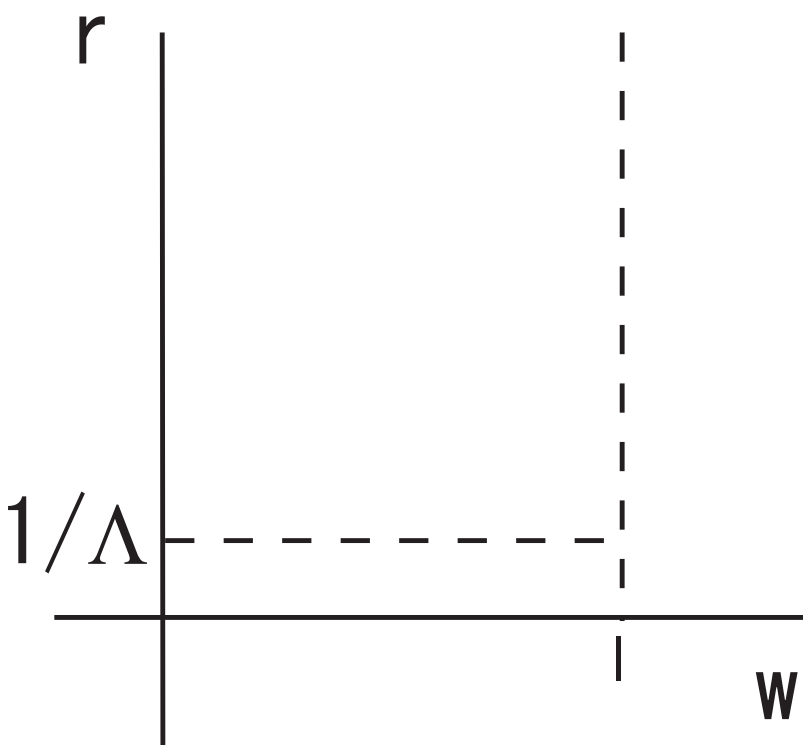


Figure 3: The integral region of (41).

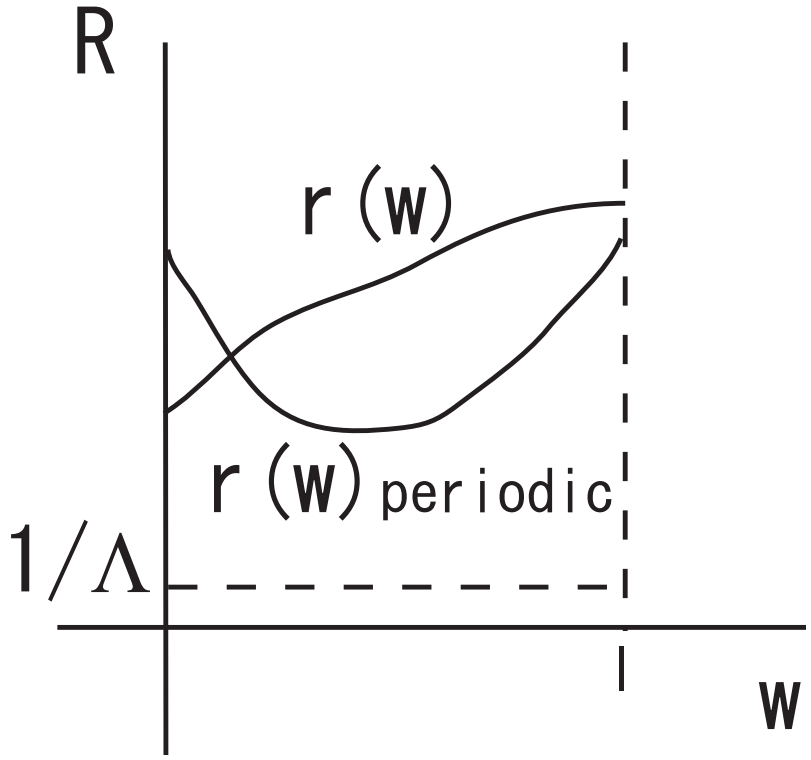


Figure 4: A general path $r(w)$ of (42) and a periodic path $r(w)$ of (43).

We regularize the above expression using the path-integral as

$$E_{Cas}^{\mathcal{W}}{}'/(2L)^2 = \frac{1}{2^2\pi^3}(2\pi)\int_{\text{all paths } r(w)}$$

$$\prod_w \mathcal{D}r(w)\left[\int dw' P(\frac{1}{r(w')})r(w')^{-4}\right]\exp\{-\mathcal{W}[r(w)]\} \quad , \tag{42}$$

where the integral is over all paths $r(w)$ which are defined between $0 \leq w \leq l$ and whose value is above Λ^{-1}, as shown in Fig.4. $\mathcal{W}[r(w)]$ is some damping functional. $\mathcal{W}[r(w)] = 0$ corresponds to (41). The slightly-more-restrictve regularization is

$$E_{Cas}^{\mathcal{W}}/(2L)^2 = \frac{1}{2^2\pi^3}(2\pi)\int_{\Lambda^{-1}}^{\infty} d\rho \int_{r(0)=r(l)=\rho}$$

$$\prod_w \mathcal{D}r(w)\left[\int dw' P(\frac{1}{r(w')})r(w')^{-4}\right]\exp\{-\mathcal{W}[r(w)]\} \quad \geq 0 \quad , \tag{43}$$

where the integral is over all periodic paths. Note that the above regularization keep the positive-definite property. Hence the present regularization mainly defined by the choice of $\mathcal{W}[r(w)]$. In order to specify it, we introduce the following metric in (R, w)-space.

$$\text{Dirac Type} \ : \quad ds^2 = dR^2 + V(R)dw^2 \quad , V(R) = \Omega^2 R^2 \ \ HarmonicOsc. \quad , \tag{44}$$

or

$$\text{Standard Type} \ : \quad ds^2 = \frac{1}{dw^2}(dR^2 + V(R)dw^2)^2 \quad , V(R) = \Omega^2 R^2 \quad . \tag{45}$$

Ω : regularization constant. (When $V(R) = 1$, w is the familiar Euclidean time.

) On a path $R = r(w)$, the induced metric and the length L is given as follows. As the damping functional $\mathcal{W}[r(w)]$, we take the length L.

$$ds^2 = dw^2(r'^2 + \Omega^2 r^2)^2 \quad , \quad r' \equiv \frac{dr}{dw} \quad ,$$

$$L = \int ds = \int (r'^2 + \Omega^2 r^2) dw \quad , \quad \mathcal{W}[r(w)] \equiv \frac{1}{2\alpha} L = \frac{1}{2\alpha} \int (r'^2 + \Omega^2 r^2) dw \quad . \tag{46}$$

α , Ω : regularization parameters. The limit $\alpha \to \infty$ corresponds to (41).

Numerical calculation can evaluate $E^{\mathcal{W}}_{Cas}$ (43), and we expect the following form[PTP121(2009)727].

$$\frac{E^{\mathcal{W}}_{Cas}}{(2L)^2} = \frac{a}{l^3}(1 - 3c \ln\,(l\Lambda)) \quad , \tag{47}$$

where a and c are some constants. a should be positive because of the positive-definiteness of (43). The present regularization result has, like the ordinary renormalizable ones such as the coupling in QED, the log-divergence. The divergence can be renormalized into the boundary parameter l. This means l flows according to the renormalization group.

$$l' = l(1 - 3c\ln(l\Lambda))^{-\frac{1}{3}} \quad , \quad \beta \equiv \frac{d\ln(l'/l)}{d\ln\Lambda} = c \quad , \quad |c| \ll 1 \quad , \tag{48}$$

where β is the renormalization group function, and we assume $|c| \ll 1$. The sign of c determines whether the length separation increases ($c > 0$) or decreases ($c < 0$) as the measurement resolution becomes finer (Λ increases). In terms of the usual terminology, attractive case corresponds to $c > 0$, and repulsive case to $c < 0$.

7. Conclusion

We have proposed a new regularization, in the quantum field theory, for the calculation of divergent physical quantities such as Casimir energy.

- $ds^2 = dR^2 + \Omega^2 R^2 dw^2$ (Elastic view to the space)

- Path integral using Hamiltonian (Weight functional) of length.

- Positive definite